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THE APPLICATION OF FINITE ELEMENT METHOD FOR MHD VISCOUS FLOW OVER A POROUS STRETCHING SHEET

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This work is concerned with the magnetohydrodynamic (MHD) viscous flow due to a porous stretching sheet. The similarity solution of the problem is obtained using finite element method. The physical quantities of interest like the fluid velocity and skin friction coefficient is obtained and discussed under the influence of suction parameter and Hartman number. It is evident from the results that MHD can be used to control the boundary layer thickness.

Keywords: MHD viscous flow, Stretching sheet, Finite element method.

1. Introduction

The boundary layer viscous flow induced by stretching surface moving with a certain velocity in an otherwise quiescent fluid medium often occurs in several engineering processes. Such flows have promising applications in industries, for example in the extrusion of a polymer sheet from a die or in the drawing of plastic films. During the manufacture of these sheets, the melt issues from a slit and is subsequently stretched to achieve the desired thickness. The mechanical properties of the final product strictly depend on the stretching and cooling rates in the process.

Since the pioneering work of Sakiadis [1,2] various aspects of boundary layer flow due to a stretching sheet have been investigated by several workers in the field. Specifically Crane's problem [3] for flow of an incompressible viscous fluid past a stretching sheet has become a classic in the literature. It admits an exact analytical solution. Besides it has produced a galore of associated problems, each incorporating a new effect and still giving an exact solution. The uniqueness of the exact analytical solution presented in [3] is discussed by McLeod and Rajagopal [4]. Gupta and Gupta [5] examined the stretching flow subject to suction or injection. The flow inside a stretching channel or tube has been analyzed by Brady and Acrivos [6] and the flow outside the stretching tube by Wang [7]. In another paper, Wang [8] extended the flow analysis to the three-dimensional axis symmetric stretching surface. The unsteady flow induced by a stretching film has been also discussed by Wang [9] and Usha and Sridharan [10].

The objective of the present paper is to obtain a numerical solution using finite element method for MHD flow over a porous stretching sheet. The paper is organized as follows:

Section 2 contains the mathematical formulation of the problem. The numerical solution using finite element method is presented in section 3. Section 4 contains the analysis of results and their discussions. In section 5 we have included some concluding remarks.

2. Mathematical Formulation

In Cartesian coordinates the continuity and momentum equations for two-dimensional MHD viscous flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B_0^2}{\rho}u, \qquad (2)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \qquad (3)$$

where $\nu=\mu/\rho$ is the kinematic viscosity and σ is the electrical conductivity. We have applied the magnetic field B_0 in the z -direction and the induced magnetic field is neglected. The above equations are derived by considering the zero electric field and incorporating the small magnetic Reynold number assumption. Under the usual boundary layer approximations the flow is

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governed by the continuity equation equation (1) and the momentum equation takes the following form :

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u, \quad (4)$$

in the above equation we have neglected the pressure gradient because the flow is caused only due to the stretching of the sheet.

The boundary conditions applicable to the present flow are

$$\begin{aligned} &u=ax, \quad v=-W \quad \text{at } y=0,\\ &u\to 0 \qquad\qquad \text{as } y\to\infty. \end{aligned} \quad (5)$$

in which $a>0$ is the stretching constant, W is the suction velocity. Defining

$$u=axf'(\eta), \qquad v=-\sqrt{a\nu}f(\eta), \qquad \eta=\sqrt{\frac{a}{\nu}}y. \quad (6)$$

Equation (1) is identically satisfied and Eqs. (4) and (5) becomes

$$f'''-M^2f'-f'^2+ff''=0, \quad (7)$$

$$\begin{aligned} &f=s, \quad f'=1 \quad \text{at } \eta=0,\\ &f'\to 0 \quad \text{as} \quad \eta\to\infty, \end{aligned} \quad (8)$$

where $s=W/m\sqrt{a\nu}$ and $M^2=\sigma B_0{}^2/\rho a$.

3. Finite Element Method

The finite element method has been employed for the solution of this non-linear differential equation. Finite element method is a well developed numerical technique for obtaining approximate solutions to a wide variety of linear or non-linear differential equations arising in engineering and mathematical physics. In finite element method, a given boundary value problem is first transformed into a weak form or variational form. A weak form is a weighted integral statement of a differential equation in which the differentiation is distributed among the dependent variable and the weight function or test function, and includes the natural boundary conditions of the problem. The weak formulation has two desirable characteristics. First, it requires weaker continuity of the dependent variable by distributing the differentiation between the solution and the weight function w (due to its weaker requirement of continuity, it has been given the name weak form). Second, the natural boundary conditions of the problem are included in the weak form, and the solution is required to satisfy only the essential boundary conditions of the problem. Whenever, the classical solution exists it coincides with the weak solution of the problem.

In this method the continuous physical model or domain is divided into finite number of smaller elements/sub-domains which is called discretization. The domain for the boundary value problem is viewed as an assemblage of these sub-domains usually known as finite element mesh/grid. The points at which these elements are connected are called nodes or nodal points. An approximate solution is then computed on these node points.

Instead of solving the problem for the entire domain in one step, attention is mainly devoted to the formulation of the properties of the constituent elements. A standard element is selected from the mesh and then finite element formulation is constructed for this element. Results are recombined to represent the whole domain/mesh. Since these elements can be put together in a variety of ways, they can be used to represent very complex domain shapes. The mesh consists of line segments in one dimension, in two dimensions it may consist of triangles or quadrilaterals and in three dimensions it may consist of tetrahedra or hexahedra. All these are known as finite elements or simply elements.

A variety of element shapes may be used and with care different element shapes may be employed in the same solution region. If we partition the domain Ω into a finite number E of elements Ω_1, $\Omega_2,\ldots,$ Ω_E, then these elements should be non overlapping and cover the domain Ω in the sense that,

$$\Omega_e\cap\Omega_f=\phi \ \text{ for } \ e\neq f \ \text{ and } \ \bigcup_{e=1}^{E}\overline{\Omega}_e=\overline{\Omega}. \quad (9)$$

The number and the type of elements to be used in a given problem are matters of mathematical or engineering judgment.

The finite element method works by expressing the unknown field variable in terms of assumed approximating functions within each element. The approximating functions (sometimes called interpolation functions) are defined in terms of linear combinations of algebraic polynomials called basis functions and the values of the field variables at node points. The nodal values of the field variable and the basis functions for the elements completely define the behavior of the field variable within the elements. For the finite element representation of a problem the nodal values of the field variable become unknowns to be determined.

A finite element approximate solution is of the type,

$$U^h = \sum_{i=1}^{N} u_i \psi_i, \tag{10}$$

where u_i are solution values at the node points to be determined and ψ_i are chosen approximating functions.

The choice of algebraic polynomials as a basis function has two reasons. First the interpolation theory of numerical analysis can be used to develop the approximate functions systematically over an element. Second, numerical evaluation of integrals of algebraic polynomials is easy. The degree of the polynomial chosen depends on the number of nodes assigned to the element, the nature and number of unknowns at each node and certain continuity requirements imposed at the nodes and along the element boundaries. For example, in two dimensions on triangles the field variables may be approximated by linear polynomials $p = \alpha_1 + \alpha_2 x + \alpha_3 y$, with three nodes at the vertices of the triangle or by quadratic polynomials $p = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$, with six nodes, three at the vertices and three at the mid points of the triangle edges. Basis functions ψ_i have the following properties.

1. The functions ψ_i are bounded and continuous, that is, $\psi_i \in C(\overline{\Omega})$.
2. The total number of basis functions is equal to the number of nodes present in the mesh and each function ψ_i is nonzero only on those elements that are connected to node i : $\psi_i(\underline{x})|_{\Omega_e} \equiv 0$ if $i \notin \overline{\Omega}_e$.
3. ψ_i is equal to 1 at node *i*, and equal to zero at the other nodes, $\psi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$

In Galerkin approach the approximate solution of original problem for an element is sought by choosing test/weight function equivalent to the basis function for that element. On substituting the approximate finite element solution in the weak form we get the algebraic element equations. This yields a large set of simultaneous algebraic equations. After imposing the essential and natural boundary conditions the problem is thus reduced to one of solving the set of simultaneous equations where the number of equations is equal to the number of nodes at which the solution is required. In matrix form this set of equations can be written as

$$K\underline{u} = \underline{f}, \tag{11}$$

where the matrix K is known as the stiffness matrix and $\underline{f}$ is known as the load vector.

Since $\psi_i = 0$ for all elements that do not have node i as a node, it follows that this property of basis functions will result in the matrix K having a sparse structure or, with an appropriate ordering, a banded structure in which all nonzero entries is clustered around the main diagonal.

3.1. Implementation of finite element method

To solve Eq. (7), with the boundary conditions given by Eq. (8), we assume

$$f' = v \tag{12}$$

Equation (7) then becomes,

$$v'' - v^2 + fv' - M^2 v = 0. \tag{13}$$

The corresponding boundary conditions then become

$$f(0) = s, \quad v(0) = 1, \quad v(\infty) = 0. \tag{14}$$

The variational form associated with Eqs. (12) and (13) over a typical two node line element (y_i, y_{i+1}) is given by

$$\int_{y_i}^{y_{i+1}} \phi_1 \left(v'' - v^2 + fv' - M^2 v \right) dy = 0, \tag{15}$$

$$\int_{y_i}^{y_{i+1}} \phi_2 \left(v - f' \right) dy = 0, \tag{16}$$

where ϕ_1 and ϕ_2 are arbitrary test functions which can be considered as the variations of v and f, respectively. We assume the finite element solution over this element is of the form,

$$\mathbf{v} = \sum_{i=1}^{2} v_i \psi_i, \quad \mathbf{f} = \sum_{i=1}^{2} f_i \psi_i, \tag{17}$$

For Galerkin finite element solution approximation we use

$$\phi_1 = \phi_2 = \psi_i \quad (i = 1..2) \tag{18}$$

Where ψ_i are the basis functions for a typical element (y_i, y_{i+1}) and they are defined as

$$\psi_1 = \frac{y_{i+1} - y}{y_{i+1} - y_i}, \quad \psi_2 = \frac{y - y_i}{y_{i+1} - y_i} \quad y_i \le y \le y_{i+1}. \tag{19}$$

Equations (15) and (16) will make a system of non linear simultaneous equations. Finite element formulation of Eqs. (15) and (16) will also generate a nonlinear set of algebraic equations. Eqs. (15) and (16) can be written in matrix form as,

$$\mathbf{G}(\mathbf{u}) = \begin{bmatrix} g_1(v,f) \\ g_2(v,f) \\ . \\ . \\ g_n(v,f) \end{bmatrix} = \mathbf{0}, \tag{20}$$

where $g_i(v,f)$ can be found by putting in the value of v and f from Eq. (17) in Eqs. (15) and (16).

The domain of the problem is divided into a set of 200 elements of equal length. By incorporating the boundary conditions we obtain a set of 399 simultaneous non linear algebraic equations having 399 unknowns. For the solution of this set of equations we used Newton's iterative method with an initial guess provided to it as

$$f(y) = 1 - \exp^{-y}. \tag{21}$$

Newton's iterative method is generally implemented in a two step procedure. First a vector $\mathbf{z}$ is found which will satisfy

$$J(\mathbf{u}^{(k)})\mathbf{z} = -\mathbf{G}(\mathbf{u}^{(k)}), \tag{22}$$

where $J(\mathbf{u}^{(k)})$ is the Jacobian of $\mathbf{G}(\mathbf{u}^{(k)})$. After this has been carried out the new approximation $\mathbf{u}^{(k+1)}$, can be obtained by

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \mathbf{z}. \tag{23}$$

Newton's method is expected to give quadratic convergence, provided that a sufficiently good starting guess is given.

4. Results and Discussion

The graphs for the function $f'(\eta)$ which corresponds to velocity component u and $f(\eta)$ that corresponds to velocity component v are drawn against η for different values of the parameters s and M are shown in Figs. 1 and 2.

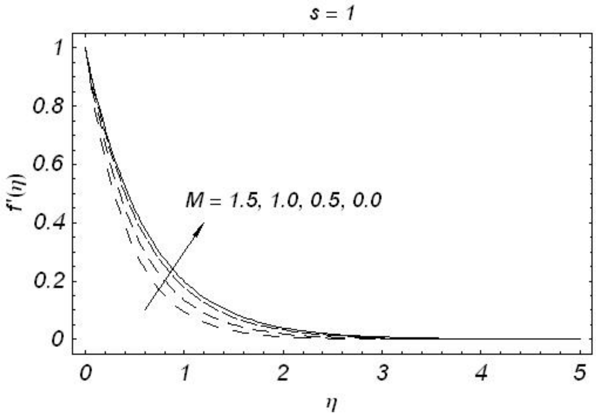

Fig. 1a

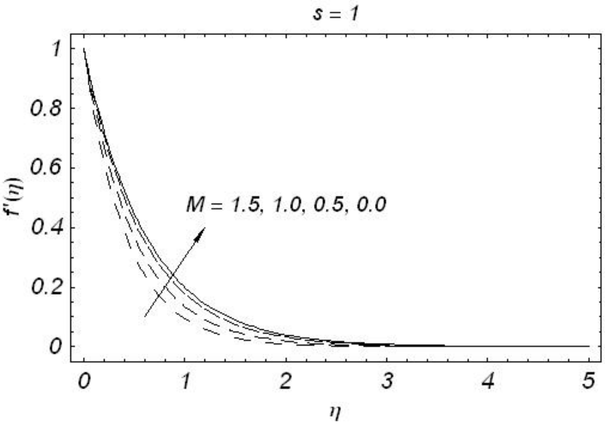

Fig. 1b

Figure 1. Influence of MHD parameter M on the velocity components f' and f.

It is shown in Fig. 2(a) that the x-component of velocity and boundary layer thickness decreases with an increase in the MHD parameter. However, the y-component of velocity decreases but boundary layer thickness increases by increasing the MHD parameter. The effects of suction parameter are quite similar to that of the MHD parameter and are shown in Figs. 2. The value of

Table 1: Variation in $f''(0)$ for different values of M and s.

s	M = 0.0	M = 0.5	M = 1.0	M = 1.5
0.0	-1.00000	-1.08750	-1.36565	-1.72447
1.0	-1.55490	-1.65316	-1.90426	-2.23727
2.0	-2.27642	-2.35246	-2.55659	-2.84256
3.0	-3.05033	-3.10817	-3.26929	-3.50621

skin friction coefficient is tabulated in table 1 under the influence of MHD and suction parameters.

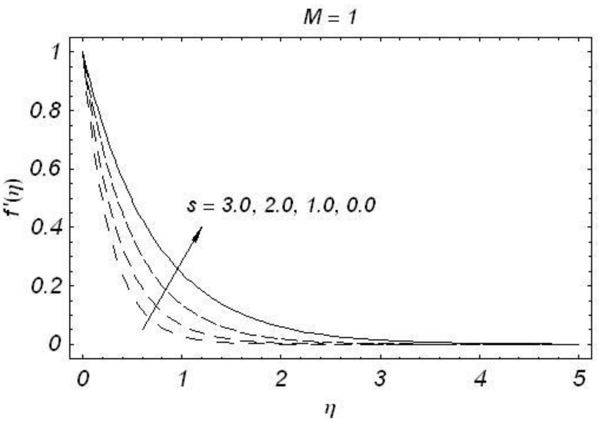

Fig. 2a

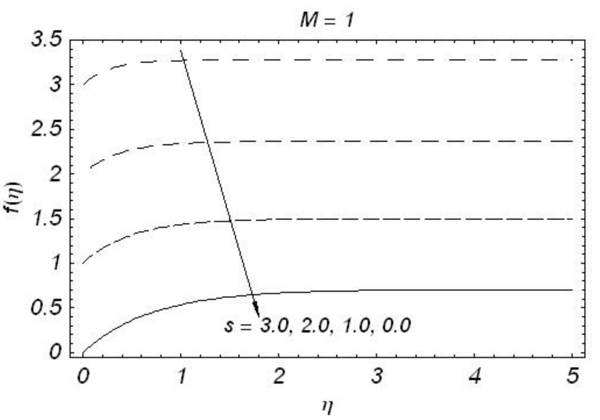

Fig. 2b

Figure 2. Influence of suction parameter S on the velocity components f' and f.

Table 1 elucidate that the magnitude of skin friction coefficient increases by increasing both MHD parameter M and suction parameter s Hence it is noted that at the surface of stretching sheet the more force is required to drag an MHD flow as compared to the hydrodynamic flow and this will cause a reduction in the boundary layer thickness. A similar phenomenon is observed for the suction velocity s.

5. Concluding Remarks

In this paper, the MHD viscous flow due to a porous stretching sheet is considered. The numerical solution is obtained using finite element method. The results are presented graphically and the effects of the emerging parameters are discussed. It is noted that the boundary layer thickness is reduced by introducing the MHD effects.

References

[1] B. C. Sakiadis, AICHE J. **7** (1961) 26.

[2] B. C. Sakiadis, AICHE J. **17** (1961) 221.

[3] L. J. Crane, Z. Angew Math. Mech., **21** (1970) 645.

[4] J. B. McLeod and K. R. Rajagopal, Arch. Rat. Mech. Anal., **98** (1987) 385.

[5] P. S. Gupta and A. S. Gupta, J. Chem. Engg. **55** (1977) 744.

[6] J. F. Brady and A. Acrivos, J. Fluid Mech., **112** (1981) 127.

[7] C. Y. Wang, Phys. Fluids., **31** (1988) 466.

[8] C. Y. Wang, Phys. Fluids., **27** (1984) 1915.

[9] C. Y. Wang, Quart. Appl. Math., **48** (1990) 601.

[10] R. Usha and R. Sridharan, J. Fluids Eng., **117** (1995) 81.