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Induced Averaging Aggregation Operators with Interval Pythagorean Trapezoidal Fuzzy Numbers and their Application to Group Decision Making

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ABSTRACT

Pythagorean fuzzy number is a new tool for uncertainty and vagueness. It is a generalization of fuzzy numbers and intuitionistic fuzzy numbers. This paper deal with induced interval Pythagorean trapezoidal fuzzy numbers. In this paper we introduce induced interval Pythagorean trapezoidal fuzzy numbers and some operation on I-IPTFN, and we also define different types of operators for aggregating induced interval Pythagorean trapezoidal fuzzy numbers. We present induced interval Pythagorean trapezoidal fuzzy ordered weighted averaging (I-IPTFOWA) operator and induced interval Pythagorean trapezoidal fuzzy hybrid averaging (I-IPTFHA) operator. Finally we develop a general algorithm for group decision making problem.

1. Introduction

The notion of fuzzy set theory was established by L.A. Zadeh [1] in 1965. In fuzzy set theory the degree of member ship function was discussed. Fuzzy set theory has been studied in various direction such that, homoeopathic verdict, computer science, fuzzy algebra and decision making problems. In 1986 Atanassov [2] present the idea of IFS, and discussed the degree of membership as well as the degree of non-membership of a set by a function. IFS is the Fuzzy set theory has been studied in various direction such that, homoeopathic verdict, computer science, fuzzy algebra and decision making problems. In1986 Atanassov [2] obtainable the idea of . IFS, and discussed the degree of membership as well as the degree of non-membership of a set by a function. IFS is the generalization of fuzzy set theory. There are many advantage of IFS theory such as using in engineering, management science, computer science [3-8]. Atanassov also presented some relation and changed mathematically operations such as, algebraic product, sum, union, intersection and complement [9, 10]. He also introduced the thought of pseudo fixed topics of all operators defined over the IFSs [11]. In 1986, many scholars [12] have complete works in the field of IFS and its presentations. Mostly, data aggregation is a very fundamental research area in IFS theory that has been accepting increasingly center. Xu and Yager [13] introduced the view of dynamic IFWA operator and developed a method to explain the dynamic intuitionistic fuzzy multi attribute decision making (MADM) problems. Xu and Chen [14] introduced some new types of aggregation operators including, interval-valued intuitionistic fuzzy hybrid averaging (IVIFHA) operator, interval-valued intuitionistic fuzzy ordered weighted averaging (IVIFOWA) operator, interval-valued intuitionistic fuzzy weighted averaging (IVIFWA) operator and also proved the importance of interval-valued intuitionistic fuzzy hybrid averaging (IVIFHA) operator to multi criteria group decision making problems under interval-valued intuitionistic fuzzy data. Furthermore in Xu and Chen [15] introduced the idea of interval-valued intuitionistic fuzzy hybrid geometric (IVIFGH) operator, interval-valued intuitionistic fuzzy ordered weighted geometric (IVIFOWG) operator, interval-valued intuitionistic fuzzy weighted geometric (IVIFWG)operator.

Like the other scholars, Wang [16] also worked in the field of intuitionistic fuzzy set and presented the knowledge of intuitionistic trapezoidal fuzzy (ITFNs) numbers and interval-valued intuitionistic trapezoidal fuzzy (IVITFNs) numbers. Wang [17] not only established the idea of these numbers but also introduced the concept of Hamming distance for TIFNs as well as introduced a series of averaging aggregation operators for ITFNs such as ITFHWA, ITFOWA and ITFWA aggregation operators. In 2013, Yager [18] also worked in the field of Pythagorean fuzzy (PFs) set and introduced the idea of PFs which is a generalization of IFSs, in which the square of their sum less than or equal to 1. Su [19] also worked in the field of aggregation operators and developed some new types aggregation operators including induced intuitionistic fuzzy hybrid averaging (I-IFHA) operator, induced intuitionistic fuzzy ordered weighted averaging (I-IFOWA)operator, induced intuitionistic fuzzy hybrid geometric (I-IFHG) operator, induced interval intuitionistic fuzzy ordered weighted averaging aggregation (I-IIFOWA) operator, induced interval-valued intuitionistic fuzzy hybrid averaging aggregation (I-IIFHA) operator and induced interval-valued intuitionistic fuzzy hybrid geometric aggregation (I-IIFHG) operator. Rahman [20, 21] worked on various types of induced Pythagorean aggregation operators.

Thus an advantage of the above mention aggregation operators we develop a series of induced interval Pythagorean trapezoidal fuzzy aggregation operators. Containing the interval-valued Pythagorean trapezoidal fuzzy weighted averaging (IPTFWA) operator, induced interval-valued Pythagorean trapezoidal fuzzy ordered weighted averaging (I-IPTFOWA) operator and the induced interval –valued Pythagorean trapezoidal fuzzy hybrid averaging (I-IPTFHA) operator.

In Section 2 we give the concept of some basic definitions and operators which will be used in our later sections. In Section 3, we develop the concept of the IPTFWA and the I-IPTFHA operators and their properties. In Section 4 we give an application of I-IPTFOWA and I-IPTFHA operators to multiple attribute group decision making (MAGDM) problems with interval Pythagorean trapezoidal fuzzy information. In Section 5 we give numerical example. Concluding remarks are made in Section 6.

2. Preliminaries

In this section we define basic definition, results and operational laws.

Definition 2.1 [2] Let a set L be fixed. An IFS U in L is an object having the form:

$$U=\{\langle x,\Psi_u(l),\Upsilon_u(l)\rangle \mid l\in L\}$$

where $\Psi_u : L\to[0,1]$ and $\Upsilon_u : L\to[0,1]$

represent the degree of membership and the degree of non-membership of the element $l\in L$ to U, respectively, and for all $l\in L$:

$$0\le \Psi_u(l)+\Upsilon_u(l)\le 1$$

For each *(IFS) U in L*

$$\pi_U(l)=1-\Psi_U(l)-\Upsilon_U(l),\ \text{for all } l\in L$$

$\pi_A(l)$ is called the degree of indeterminacy of l *to* U.

Definition 2.2 [17] Let p be trapezoidal fuzzy number, its membership function

$$\Psi_p(l)=\begin{cases}\frac{x-p}{q-p}\Psi_p, & p\le l\le q;\\ \Psi_p, & q\le l\le r;\\ \frac{s-x}{s-r}\Psi_p, & r\le l\le s;\\ 1, & \text{otherwise,}\end{cases} \quad (1)$$

Its non-membership function is

$$\Upsilon_p(l)=\begin{cases}\frac{s-l+\Upsilon_p(l-p_1)}{q-p} & p_1\le l\le q;\\ \Upsilon_p, & q\le l\le r;\\ \frac{l-r+\Upsilon_p(s_1-l)}{s_1-r}, & r\le l\le s_1;\\ 0, & \text{otherwise,}\end{cases} \quad (2)$$

Where $0\le\Psi_{\tilde{\alpha}}\le 1;\quad 0\le\Upsilon_{\tilde{\alpha}}\le 1;$

$$0\le(\Psi_{\tilde{\alpha}})+(\Upsilon_{\tilde{\alpha}})\le 1;\ p,q,r,s\in R.$$

Then,

$$\tilde{p}=\langle([p,q,r,s];\Psi_{\tilde{\alpha}}),([p_1,q,r,s_1];\Upsilon_{\tilde{\alpha}})\rangle$$

is called trapezoidal fuzzy number. For convenience,

$$\tilde{p}=([p,q,r,s];\Psi_{\tilde{\alpha}},\Upsilon_{\tilde{\alpha}}).$$

Definition 2.3 [20] An OWA operator of n dimension is a mapping, OWA: $\Omega^n\to\Omega$ hat has an associated vector $\mu=(\mu_1,\mu_2,...,\mu_n)^T$ with $\mu_j\in[0,1]$ and $\Sigma_{j=1}^n\mu_j=1.$ Furthermore

$$OWA(\tilde{\alpha}_1,\tilde{\alpha}_2,...,\tilde{\alpha}_n)=\sum_{j=1}^n\mu_jC_j$$

C_j is the largest of $\tilde{\alpha}_j$.

Definition 2.4 [21] an $I-OWA$ operator is defined as follows:

$$IOWA(\langle u_1,\tilde{\alpha}_1\rangle,\langle u_2,\tilde{\alpha}_2\rangle,...,\langle u_n,\tilde{\alpha}_n\rangle)=\sum_{j=1}^n\mu_jC_j$$

where $\mu=(\mu_1,\mu_2,...,\mu_n)^T$ is the weighted vector with conditions, $\mu_j\in[0,1]$ and $\Sigma_{j=1}^n\mu_j=1.$ Then C_j is $\tilde{\alpha}_j$ value of OWA pair $\langle\mu_j,\tilde{\alpha}_j\rangle$ having the jth largest μ_j which is referred to as the order inducing variable

Definition 2.5 [17] Let $\tilde{p}_1=([p_1,q_1,r_1,s_1];\Psi_{\tilde{\alpha}_1},\Upsilon_{\tilde{\alpha}_1}),$ and $\tilde{p}_2=([p_2,q_2,r_2,s_2];\ \Psi_{\tilde{\alpha}_2},\ \Upsilon_{\tilde{\alpha}_2}),$ are two trapezoidal fuzzy numbers, and $\delta\ge 0$.

Then

(1) $\tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [p_1+p_2, q_1+q_2, r_1+r_2, s_1+s_2]; \\ (\Psi_{\tilde{\alpha}_1})+(\Psi_{\tilde{\alpha}_2})-(\Psi_{\tilde{\alpha}_1}\Psi_{\tilde{\alpha}_2}), \Upsilon_{\tilde{\alpha}_1}\Upsilon_{\tilde{\alpha}_2} \end{pmatrix};$

(2) $\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [p_1p_2, q_1q_2, r_1r_2, s_1s_2]; \Psi_{\tilde{\alpha}_1}\Psi_{\tilde{\alpha}_2}, \\ (\Upsilon_{\tilde{\alpha}_1})+(\Upsilon_{\tilde{\alpha}_2})-(\Upsilon_{\tilde{\alpha}_1}\Upsilon_{\tilde{\alpha}_2}) \end{pmatrix};$

(3) $\delta\tilde{p} = ([\delta p, \delta q, \delta r, \delta s];\ 1-(1-\Psi_{\tilde{\alpha}})^{\delta}; (\Upsilon_{\tilde{\alpha}})^{\delta});$

(4) $\tilde{p}^{\delta} = ([p^{\delta}, q^{\delta}, r^{\delta}, s^{\delta}];\ \Psi_{\tilde{\alpha}}^{\delta}, 1-(1-\Upsilon_{\tilde{\alpha}})^{\delta}).$

Example 2.6: Let

$\tilde{p} = ([0.5, 0.4, 0.6, 0.9];\ 0.3, 0.5),$
$\tilde{p}_1 = ([0.3, 0.4, 0.5, 0.3];\ 0.4, 0.6),$
$\tilde{p}_2 = ([0.4, 0.5, 0.4, 0.3];\ 0.5, 0.4)$

are trapezoidal fuzzy numbers and $\delta = 0.5$ Then we verify the above results such that,

(1) $\tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [0.3+0.4, 0.4+0.5, 0.5+0.4, 0.3+0.3]; \\ (0.4)+(0.5)-(0.4)(0.5), (0.6)(0.4) \end{pmatrix}$
$= ([0.7, 0.9, 1, 0.6]; 0.7, 0.24).$

(2) $\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [(0.3)(0.4), (0.4)(0.5), (0.5)(0.4), (0.3)(0.3)]; \\ (0.4)(0.5), (0.6+0.4)-(0.6)(0.4) \end{pmatrix}$
$= ([0.12, 0.2, 0.2, 0.9]; 0.2, 0.76).$

(3) $\delta\tilde{p} = \begin{pmatrix} [(0.5)(0.5), (0.5)(0.4), (0.5)(0.6), (0.5)(0.9)]; \\ (1-(1-0.3)^{0.5}(0.5)^{0.5} \end{pmatrix}$
$= ([0.25, 0.2, 0.3, 0.45]; 0.16, 0.70).$

(4) $\tilde{p}^{\delta} = \begin{pmatrix} [(0.5)^{0.5}, (0.4)^{0.5}, (0.6)^{0.5}, (0.9)^{0.5}]; \\ (0.3)^{0.5} 1-(1-0.5)^{0.5} \end{pmatrix};$
$= ([0.70, 0.63, 0.85, 0.94]; 0.59, 0.29).$

Definition 2.7: [18] Let *L* be a fixed set. The PFS *U* in *L* is the thing taking the shape :

$$A = \{\langle l, \Psi_U(l), \Upsilon_U(l)\rangle \mid l \in L\}$$

where $\Psi_A : L \to [0,1]$ and $\Upsilon_A : L \to [0,1]$ represent the degree of membership and the degree of non-membership of the element $l \in L$ to *A*, respectively, and for every $l \in L$:

$$0 \le \Psi_U \le 1, 0 \le \Upsilon_U \le 1, 0 \le \Psi_U^2(l) + \Upsilon_U^2(l) \le 1$$

For each $(PFS)U$ in *L*

$\pi_U(l) = \sqrt{1-\Psi_U^2(l)-\Upsilon_U^2(l)},$ for all $l \in L$

$\pi_U(l)$ is called the degree of indeterminacy of l to U.

Definition 2.8 [22] Let $\tilde{p} = (\Psi_{\alpha}, \Upsilon_{\alpha}),\ \tilde{p}_1 = (\Psi_{\alpha_1}, \Upsilon_{\alpha_1})$ and $\tilde{p}_2 = (\Psi_{\alpha_2}, \Upsilon_{\alpha_2})$ be three (PFN_s) and $\delta > 0.$

Then,

1. $\tilde{p}^c = (\Upsilon_{\alpha}, \Psi_{\alpha});$
2. $\tilde{p}_1 \oplus \tilde{p}_2 = \left(\sqrt{(\Psi_{\tilde{\alpha}_1})^2+(\Psi_{\tilde{\alpha}_2})^2-(\Psi_{\tilde{\alpha}_1}^2\Psi_{\tilde{\alpha}_2}^2)}, \Upsilon_{\tilde{\alpha}_1}\Upsilon_{\tilde{\alpha}_2}\right);$
3. $\tilde{p}_1 \otimes \tilde{p}_2 = \left(\Psi_{\tilde{\alpha}_1}\Psi_{\tilde{\alpha}_2}, \sqrt{(\Upsilon_{\tilde{\alpha}_1})^2+(\Upsilon_{\tilde{\alpha}_2})^2-(\Upsilon_{\tilde{\alpha}_1}^2\Upsilon_{\tilde{\alpha}_2}^2)}\right);$
4. $\delta\tilde{p} = \sqrt{1-(1-\Psi_{\tilde{\alpha}}^2)^{\delta}}; (\Upsilon_{\tilde{\alpha}})^{\delta});$
5. $\tilde{p}^{\delta} = (\Psi_{\tilde{\alpha}}^{\delta}, \sqrt{1-(1-\Upsilon_{\tilde{\alpha}}^2)^{\delta}}).$

Example 2.9: Let $\tilde{p} = (0.5, 0.4), \tilde{p}_1 = (0.6, 0.3),$ $\tilde{p}_2 = (0.2, 0.4)$ are Pythagorean fuzzy numbers and $\delta = 0.6$, then we verify the above results such that,

2. $\tilde{p}_1 \oplus \tilde{p}_2 = \left(\sqrt{(0.5)^2+(0.2)^2-(0.5)^2(0.2)^2}, (0.3)(0.4)\right)$
$= (0.52, 0.12).$

3. $\tilde{p}_1 \otimes \tilde{p}_2 = \left((0.6)(0.2), \sqrt{(0.3)^2+(0.2)^2-(0.3)^2(0.2)^2}\right)$
$= (0.12, -0.0012).$

4. $\delta\tilde{p} = \left(\sqrt{(1-(1-0.5^2)^{0.6}}, (0.4)^{0.6}\right)$
$= (039, 0.57).$

5. $\tilde{p}^{\delta} = \left((0.5)^{0.6}, \sqrt{(1-(1-0.4^2)^{0.6}}\right)$
$= (065, 0.31).$

Definition 2.10: Let

$\tilde{p} = ([p, q, r, s]; \Psi, \Upsilon) = ([p, q, r, s]; [\underline{\Psi}, \overline{\Psi}], [\underline{\Upsilon}, \overline{\Upsilon}])$ be an interval Pythagorean trapezoidal fuzzy number, where $\Psi = [\underline{\Psi}, \overline{\Psi}]$ and $\Upsilon = [\underline{\Upsilon}, \overline{\Upsilon}]$ represent an interval, hence $\Psi \subset [0,1]$ and $\Upsilon \subset [0,1]$, such that $0 \le \Psi^2 + \Upsilon^2 \le 1.$

Definition 2.11: Let $\tilde{p}_1 = ([p_1, q_1, r_1, s_1];\ [\underline{\Psi}_1, \overline{\Psi}_1], [\underline{\Upsilon}_1, \overline{\Upsilon}_1]),$ and $\tilde{p}_2 = ([p_2, q_2, r_2, s_2]; [\underline{\Psi}_1, \overline{\Psi}_2], [\underline{\Upsilon}_2, \overline{\Upsilon}_2]),$ be two $I-PTF$, numbers, and $\delta \ge 0.$ Then,

(1) $\tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [p_1+p_2, q_1+q_2, r_1+r_2, s_1+s_2]; \\ \left[\sqrt{(\underline{\Psi}_1)^2+(\underline{\Psi}_2)^2-(\underline{\Psi}_1\underline{\Psi}_2)^2}, \underline{\Upsilon}_1\underline{\Upsilon}_2\right] \\ , \left[\sqrt{(\overline{\Psi}_1)^2+(\overline{\Psi}_2)^2-(\overline{\Psi}_1\overline{\Psi}_2)^2}, \overline{\Upsilon}_1\overline{\Upsilon}_2\right] \end{pmatrix};$

(2) $$\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [p_1p_2, q_1q_2, r_1r_2, s_1s_2]; \\ \left[\underline{\Psi}_1\underline{\Psi}_2, \sqrt{\underline{\Upsilon}_1^2+\underline{\Upsilon}_2^2-(\underline{\Upsilon}_1\underline{\Upsilon}_2)^2}\right], \\ \left[\bar{\Psi}_1\bar{\Psi}_2, \sqrt{\bar{\Upsilon}_1^2+\bar{\Upsilon}_2^2-(\bar{\Upsilon}_1\bar{\Upsilon}_2)^2}\right] \end{pmatrix};$$

(3) $$\delta\tilde{p} = \begin{pmatrix} [\delta p, \delta q, \delta r, \delta s]; \left[\sqrt{1-\left(1-\underline{\Psi}_{\tilde{\alpha}}^2\right)^{\delta}}, (\underline{\Upsilon}_{\tilde{\alpha}})^{\delta}\right], \\ \left[\sqrt{1-\left(1-\bar{\Psi}_{\tilde{\alpha}}^2\right)^{\delta}}, (\bar{\Upsilon}_{\tilde{\alpha}})^{\delta}\right] \end{pmatrix};$$

(4) $$\tilde{p}^{\delta} = \begin{pmatrix} \left[p^{\delta}, q^{\delta}, r^{\delta}, s^{\delta}\right]; \left[\underline{\Psi}_{\tilde{\alpha}}^{\delta}, \sqrt{1-\left(1-\underline{\Upsilon}_{\tilde{\alpha}}^2\right)^{\delta}}\right], \\ \left[\bar{\Psi}_{\tilde{\alpha}}^{\delta}, \sqrt{1-\left(1-\bar{\Upsilon}_{\tilde{\alpha}}^2\right)^{\delta}}\right] \end{pmatrix}.$$

Example 2.12: Let

$$\tilde{p} = ([0.3, 0.4, 0.5, 0.6]; [0.7, 0.2], [0.5, 0.2]),$$
$$\tilde{p}_1 = ([0.3, 0.4, 0.5, 0.6]; [0.8, 0.5], [0.6, 0.4]),$$
$$\tilde{p}_2 = ([0.5, 0.3, 0.4, 0.4]; [0.8, 0.4], [0.8, 0.3])$$

are Pythagorean trapezoidal fuzzy numbers and $\delta = 0.4$ Then we verify the above results such that,

(1) $$\tilde{p}_1 \oplus \tilde{p}_2 = \begin{pmatrix} [0.3+0.5, 0.4+0.3, 0.5+0.4, 0.6+0.4]; \\ \left[\sqrt{(0.8)^2+(0.8)^2-(0.8)^2(0.8)^2}, (0.6)(0.8)\right], \\ \left[\sqrt{(0.5)^2+(0.4)^2-(0.5)^2(0.4)^2}, (0.4)(0.3)\right] \end{pmatrix}$$
$$= ([0.8, 0.7, 0.9, 0.1]; [0.8, 0.48], [0.45, 0.12]).$$

(2) $$\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [(0.8)(0.5), (0.4)(0.3), (0.5)(0.4), (0.6)(0.4)]; \\ \left[(0.8)(0.8), \sqrt{(0.6)^2+(0.8)^2-(0.6)(0.8)}\right], \\ \left[(0.5)(0.4), \sqrt{(0.4)^2+(0.3)^2-(0.4)(0.3),}\right] \end{pmatrix}$$
$$= ([0.15, 0.12, 0.2, 0.24]; [.64, .72], [.2, .36]).$$

(3) $$\delta\tilde{p} = \begin{pmatrix} [(0.4)(0.3), (0.4)(0.4), (0.4)(0.5), (0.4)(0.6)]; \\ \left[\sqrt{(1-\left(1-0.7^2\right)^{0..4}}, (0.5)^{0.4}\right], \\ \left[\sqrt{(1-\left(1-0.2^2\right)^{0..4}}, (0.2)^{0.4}\right] \end{pmatrix}$$
$$= ([0.12, 0.16, 0.2, 0.24]; [.48, .75], [.24, .52]).$$

(4) $$\tilde{p}^{\delta} = \begin{pmatrix} \left[(0.3)^{0.4}, (0.4)^{0.4}, (0.5)^{0.4}, (0.6)^{0.4}\right]; \\ \left[, (0.7)^{0.4}\sqrt{(1-\left(1-0.5^2\right)^{0..4}}\right], \\ \left[, (0.2)^{0.4}\sqrt{(1-\left(1-0.2^2\right)^{0..4}}\right] \end{pmatrix};$$
$$= ([0.61, 0.69, 0.75, 0.85]; [0.86, 0.32], [0.52, 0.40]).$$

Definition 2.13: Let $\tilde{p} = ([p, q, r, s];\ [\underline{\Psi}, \bar{\Psi}], [\underline{\Upsilon}, \bar{\Upsilon}])$ be an interval Pythagorean trapezoidal fuzzy numbers, a score function S can be defined as follows:

$$s(\tilde{p}) = \begin{pmatrix} \dfrac{p+q+r+s}{4} \cdot \\ \dfrac{\underline{\Psi}^2 - \underline{\Upsilon}^2 + \bar{\Psi}^2 - \bar{\Upsilon}^2}{2} \end{pmatrix} \quad (3)$$

Where S(P) ∈ [-1,1]

Example 2.14: Let $\tilde{p} = ([0.8, 0.6, 0.5, 0.7];\ [0.7, 0.5],\ [0.8, 0.6])$ is Pythagorean trapezoidal fuzzy numbers, Then we verify the above results such that,

$$s(\tilde{p}) = \begin{pmatrix} \dfrac{0.8+0.6+0.5+0.7}{4} \cdot \\ \dfrac{0.7^2 - 0.8^2 + 0.5^2 - 0.1^2}{2} \end{pmatrix}$$
$$= -0.0845$$

Definition 2.15: Let $\tilde{p} = ([p, q, r, s]; [\underline{\Psi}, \bar{\Psi}], [\underline{\Upsilon}, \bar{\Upsilon}])$ be an interval Pythagorean trapezoidal fuzzy numbers, an accuracy function H can be defined as follows:

$$H(\tilde{p}) = \begin{pmatrix} \dfrac{p+q+r+s}{4} \cdot \\ \dfrac{\underline{\Psi}^2 + \underline{\Upsilon}^2 + \bar{\Psi}^2 + \bar{\Upsilon}^2}{2} \end{pmatrix} \quad H(\tilde{p}) \in [0,1]. \quad (4)$$

to determine the degree of an accuracy of the interval Pythagorean trapezoidal fuzzy numbers $\tilde{p}$, where $H(\tilde{p}) \in [0,1]$. The bigger the estimation of $H(\tilde{p})$ the further level of accuracy of the interval Pythagorean trapezoidal fuzzy numbers $\tilde{p}$.

Example 2.16: Let $\tilde{p} = ([0.8, 0.6, 0.5, 0.7];\ [0.7, 0.5],\ [0.8, 0.6])$ is Pythagorean trapezoidal fuzzy numbers, Then we verify the above results such that,

$$H(\tilde{p}) = \begin{pmatrix} \dfrac{0.8+0.6+0.5+0.7}{4} \cdot \\ \dfrac{0.7^2 + 0.8^2 + 0.5^2 + 0.1^2}{2} \end{pmatrix}$$
$$= 0.56.$$

Theorem 2.17: Let $\tilde{p}_1 = ([p_1, q_1, r_1, s_1];\ [\underline{\Psi}_1, \bar{\Psi}_1],\ [\underline{\Upsilon}_1, \bar{\Upsilon}_1])$ and $\tilde{p}_2 = ([p_2, q_2, r_2, s_2];\ [\underline{\Psi}_2, \bar{\Psi}_2],\ [\underline{\Upsilon}_2, \bar{\Upsilon}_2])$ be two $I-PTF$ numbers and $\delta, \delta_1, \delta_2$ are any scalar numbers. Then

(1) $\tilde{p}_1 \otimes \tilde{p}_2 = \tilde{p}_2 \otimes \tilde{p}_1;$

(2) $\left(\tilde{p}_1 \otimes \tilde{p}_2\right)^{\delta} = \tilde{p}_2^{\delta} \otimes \tilde{p}_1^{\delta};$

(3) $\tilde{p}^{\delta_1} \otimes \tilde{p}^{\delta_2} = \tilde{p}^{(\delta_1+\delta_2)}$.

Proof: (1) $\Pr oof$ is easy.

(2) Using definition 7 and operational law 2, we have

$$\tilde{p}_1 \otimes \tilde{p}_2 = \begin{pmatrix} [p_1p_2, q_1q_2, r_1r_2, s_1s_2]; \\ \left[\underline{\Psi}_1\underline{\Psi}_2, \sqrt{\underline{\Upsilon}_1^2 + \underline{\Upsilon}_2^2 - (\underline{\Upsilon}_1\underline{\Upsilon}_2)^2}\right] \\ ,\left[\overline{\Psi}_1\overline{\Psi}_2, \sqrt{\overline{\Upsilon}_1^2 + \overline{\Upsilon}_2^2 - (\overline{\Upsilon}_1\overline{\Upsilon}_2)^2}\right] \end{pmatrix}$$

Then by operational law (4) in Definition (7), it follows that

$$(\tilde{p}_1 \otimes \tilde{p}_2)^{\delta} = \begin{pmatrix} \left[(p_1p_2)^{\delta}, (q_1q_2)^{\delta}, (r_1r_2)^{\delta}, (s_1s_2)^{\delta}\right]; \\ \left[(\underline{\Psi}_1\underline{\Psi}_2)^{\delta}, \sqrt{1-(1-(\underline{\Upsilon}_1^2 + \underline{\Upsilon}_2^2 - (\underline{\Upsilon}_1\underline{\Upsilon}_2)^2)^{\delta}}\right] \\ ,\left[\left(\overline{\Psi}_1\overline{\Psi}_2\right)^{\delta}, \sqrt{1-(1-(\overline{\Upsilon}_1^2 + \overline{\Upsilon}_2^2 - (\overline{\Upsilon}_1\overline{\Upsilon}_2)^2)^{\delta}}\right] \end{pmatrix}$$

Also since

$$(\tilde{p}_1)^{\delta} = \begin{pmatrix} \left[(p_1)^{\delta}, (q_1)^{\delta}, (r_1)^{\delta}, (s_1)^{\delta}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_1}\right)^{\delta}, \sqrt{(1-(1-\underline{\Upsilon}_{\tilde{\alpha}_1}^2)^{\delta}}\right] \\ ,\left[\left(\overline{\Psi}_{\tilde{\alpha}_1}\right)^{\delta}, \sqrt{(1-(1-\overline{\Upsilon}_{\tilde{\alpha}_1}^2)^{\delta}}\right] \end{pmatrix}$$

$$(\tilde{p}_2)^{\delta} = \begin{pmatrix} \left[(p_2)^{\delta}, (q_2)^{\delta}, (r_2)^{\delta}, (s_2)^{\delta}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_2}\right)^{\delta}, \sqrt{(1-(1-\underline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta}}\right] \\ ,\left[\left(\overline{\Psi}_{\tilde{\alpha}_2}\right)^{\delta}, \sqrt{(1-(1-\overline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta}}\right] \end{pmatrix}.$$

Then, we have

$$(\tilde{p}_2)^{\delta} \otimes (\tilde{p}_1)^{\delta} = \begin{pmatrix} \left[(p_1p_2)^{\delta}, (q_1q_2)^{\delta}, (r_1r_2)^{\delta}, (s_1s_2)^{\delta}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_1}\underline{\Psi}_{\tilde{\alpha}_2}\right)^{\delta}, \sqrt{\begin{matrix}(1-(1-\underline{\Upsilon}_{\tilde{\alpha}_1}^2)^{\delta} + (1-(1-\underline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta} - \\ (1-(1-\underline{\Upsilon}_{\tilde{\alpha}_1}^2)^{\delta}(1-(1-\underline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta}\end{matrix}}\right] \\ ,\left[\left(\overline{\Psi}_{\tilde{\alpha}_1}\overline{\Psi}_{\tilde{\alpha}_2}\right)^{\delta}, \sqrt{\begin{matrix}(1-(1-\overline{\Upsilon}_{\tilde{\alpha}_1}^2)^{\delta} + (1-(1-\overline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta} - \\ (1-(1-\overline{\Upsilon}_{\tilde{\alpha}_1}^2)^{\delta}(1-(1-\overline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta}\end{matrix}}\right] \end{pmatrix}$$

.

$$= \begin{pmatrix} \left[(p_1p_2)^{\delta}, (q_1q_2)^{\delta}, (r_1r_2)^{\delta}, (s_1s_2)^{\delta}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}_1}\underline{\Psi}_{\tilde{\alpha}_2}\right)^{\delta}, \sqrt{(1-(1-\underline{\Upsilon}_{\tilde{\alpha}_1}^2 + \underline{\Upsilon}_{\tilde{\alpha}_2}^2 - \underline{\Upsilon}_{\tilde{\alpha}_1}^2\underline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta}}\right] \\ ,\left[\left(\overline{\Psi}_{\tilde{\alpha}_1}\overline{\Psi}_{\tilde{\alpha}_2}\right)^{\delta}, \sqrt{(1-(1-\overline{\Upsilon}_{\tilde{\alpha}_1}^2 + \overline{\Upsilon}_{\tilde{\alpha}_2}^2 - \overline{\Upsilon}_{\tilde{\alpha}_1}^2\overline{\Upsilon}_{\tilde{\alpha}_2}^2)^{\delta}}\right] \end{pmatrix}$$

Hence, $(\tilde{p}_1 \otimes \tilde{p}_2)^{\delta} = \tilde{p}_2^{\delta} \otimes \tilde{p}_1^{\delta}$.

(3) By the operational law (4) in Definition (7), we obtain

$$(\tilde{p})^{\delta_1} = \begin{pmatrix} \left[(p)^{\delta_1}, (q)^{\delta_1}, (r)^{\delta_1}, (s)^{\delta_1}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}}\right)^{\delta_1}, \sqrt{(1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1}}\right], \\ \left[(\overline{\Psi}_{\tilde{\alpha}})^{\delta_1}, \sqrt{(1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1}}\right] \end{pmatrix}$$

$$(\tilde{p})^{\delta_2} = \begin{pmatrix} \left[(p)^{\delta_2}, (q)^{\delta_2}, (r)^{\delta_2}, (s)^{\delta_2}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}}\right)^{\delta_2}, \sqrt{(1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2}}\right], \\ \left[(\overline{\Psi}_{\tilde{\alpha}})^{\delta_2}, \sqrt{(1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2}}\right] \end{pmatrix}.$$

Then,

$$(\tilde{p})^{\delta_1} \otimes (\tilde{p})^{\delta_2} = \begin{pmatrix} \left[p^{\delta_1}p^{\delta_2}, q^{\delta_1}q^{\delta_2}, r^{\delta_1}r^{\delta_2}, s^{\delta_1}s^{\delta_2}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}}\right)^{\delta_1}\left(\underline{\Psi}_{\tilde{\alpha}}\right)^{\delta_2}, \sqrt{\begin{matrix}(1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1}) + (1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2}) \\ -(1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1})(1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2})\end{matrix}}\right], \\ \left[\left(\overline{\Psi}_{\tilde{\alpha}}\right)^{\delta_1}\left(\overline{\Psi}_{\tilde{\alpha}}\right)^{\delta_2}, \sqrt{\begin{matrix}(1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1}) + (1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2}) \\ -(1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1})(1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2})\end{matrix}}\right] \end{pmatrix}$$

$$= \begin{pmatrix} \left[p^{\delta_1}p^{\delta_2}, q^{\delta_1}q^{\delta_2}, r^{\delta_1}r^{\delta_2}, s^{\delta_1}s^{\delta_2}\right]; \\ \left[\left(\underline{\Psi}_{\tilde{\alpha}}\right)^{\delta_1}\left(\underline{\Psi}_{\tilde{\alpha}}\right)^{\delta_2}, \sqrt{(1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1})(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2})}\right], \\ \left[\left(\overline{\Psi}_{\tilde{\alpha}}\right)^{\delta_1}\left(\overline{\Psi}_{\tilde{\alpha}}\right)^{\delta_2}, \sqrt{(1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1})(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_2})}\right] \end{pmatrix}$$

$$= \begin{pmatrix} \left[p^{\delta_1+\delta_2}, q^{\delta_1+\delta_2}, r^{\delta_1+\delta_2}, s^{\delta_1+\delta_2}\right]; \\ \left[\underline{\Psi}_{\tilde{\alpha}}^{\delta_1+\delta_2}, \sqrt{(1-(1-\underline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1+\delta_2}}\right], \\ \left[\overline{\Psi}_{\tilde{\alpha}}^{\delta_1+\delta_2}, \sqrt{(1-(1-\overline{\Upsilon}_{\tilde{\alpha}}^2)^{\delta_1+\delta_2}}\right] \end{pmatrix}$$

$$= (\tilde{p})^{\delta_1+\delta_2}.$$

3. Induced averaging aggregation operators with Interval-valued Pythagorean trapezoidal fuzzy numbers

In this section, we introduce the notion of interval Pythagorean trapezoidal fuzzy weighted averaging (IPTFWA) operator, induced interval Pythagorean trapezoidal fuzzy ordered weighted averaging $(I-IPTFOWA)$ operator, and interval Pythagorean trapezoidal fuzzy hybrid averaging $(I-IPTFHA)$ operator. We also discuss several properties of these operators, including idem potency, bounded, and monotonicity as follows.

Definition 3.1: Let $\tilde{p}_{\pounds}\left(j=\pounds=1,2,...,\Phi\right)$ be a group of *IPTF* numbers, let Ω be set of *IPTF* numbers, such that *IPTFWA*, $\Omega^{\Phi}\rightarrow\Omega$, if

$$IPTFWA\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_{\Phi}\right)=\left(\hbar_1\tilde{p}_1\oplus\hbar_2\tilde{p}_2...\oplus\hbar_{\Phi}\tilde{p}_{\Phi}\right). \quad (5)$$

Then, $IPTFWA$ is said interval Pythagorean trapezoidal fuzzy weighted averaging operator of dimension Φ. Especially, if $\hbar=\left(\hbar_1,\hbar_2,...,\hbar_{\Phi}\right)^T$ having weight such that $\tilde{p}_{\pounds}\left(j=\pounds=1,2,...,\Phi\right)$ *with* $\hbar_{\pounds}\in\left[0,1\right]$ *and* $\Sigma_{\pounds=1}^{\Phi}\hbar_{\pounds}=1$, *if* $\hbar=\left(\frac{1}{\Phi},\frac{1}{\Phi},...,\frac{1}{\Phi}\right)^T$. *Then* *IPTFWA* operator is reduced to interval Pythagorean trapezoidal fuzzy averaging (*IPTFA*) operator of measurement Φ which is characterized as takes after :

$$IPTFA_w\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_{\Phi}\right)=(\tilde{p}_1\oplus\tilde{p}_2...\oplus\tilde{p}_{\Phi})^{\frac{1}{\Phi}} \quad (6)$$

By Definition 12 and Theorem 1 , we can acquire the accompanying outcome. In order to proof, we use mathematical induction.

Theorem 3.2: Consider $\tilde{p}_{\pounds}\left(j=\pounds=1,2,...,\Phi\right)$ is the group of *IPTF* numbers. At that point, they collected an incentive by utilizing the *IPTFWA* administrator is an additionally *IPTF* number with the end goal that

$$IPTFWA(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_{\Phi})=\begin{pmatrix}\left[\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}p_{\pounds},\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}q_{\pounds},\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}r_{\pounds},\sum_{\pounds=1}^{\Phi}\hbar_{\pounds}s_{\pounds}\right];\\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_{\pounds}}\right)^{\hbar_{\pounds}}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\overline{\Psi}^2_{\tilde{\alpha}_{\pounds}}\right)^{\hbar_{\pounds}}}\right],\\ \left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}_{\pounds}^{\hbar_{\pounds}},\prod_{\pounds=1}^{\Phi}\overline{\Upsilon}_{\pounds}^{\hbar_{\pounds}}\right]\end{pmatrix} \quad (7)$$

where $\hbar=\left(\frac{1}{\Phi},\frac{1}{\Phi},...,\frac{1}{\Phi}\right)^T$ is the weight vector of $\tilde{p}_{\pounds}\left(\pounds=1,2,...,\Phi\right)$ with $\hbar_{\pounds}\in\left[0,1\right]$ and $\Sigma_{\pounds=1}^{\Phi}\hbar_{\pounds}=1$.

Proof: The result first is follows from Definition 12 and Theorem 1, by mathematical induction we prove the second result, we show that Eq. (7) satisfy the condition when $\Phi=2$.

$$\hbar_1\tilde{p}_1=\begin{pmatrix}\left[\hbar_1p_1,\hbar_1q_1,\hbar_1r_1,\hbar_1s_1\right];\\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\overline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}}\right],\\ \left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}^{\hbar_1}_{\tilde{\alpha}_1},\prod_{\pounds=1}^{\Phi}\overline{\Upsilon}^{\hbar_1}_{\tilde{\alpha}_1}\right]\end{pmatrix}$$

$$\hbar_2\tilde{p}_2=\begin{pmatrix}\left[\hbar_2p_2,\hbar_2q_2,\hbar_2r_2,\hbar_2s_2\right];\\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\overline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}}\right],\\ \left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}^{\hbar_2}_{\tilde{\alpha}_2},\prod_{\pounds=1}^{\Phi}\overline{\Upsilon}^{\hbar_2}_{\tilde{\alpha}_2}\right]\end{pmatrix}.$$

Then,

$$IPTFWA(\tilde{p}_1,\tilde{p}_2)=\hbar_1\tilde{p}_1\oplus\hbar_2\tilde{p}_2$$

$$=\begin{pmatrix}\left[\hbar_1p_1\hbar_2p_2,\hbar_1q_1\hbar_2q_2,\hbar_1r_1\hbar_2r_2,\hbar_1s_1\hbar_2s_2\right];\\ \left[\sqrt{\begin{matrix}1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}+1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}\\-(1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1})(1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2})\end{matrix}},\left(\underline{\Upsilon}^{\hbar_1}_{\tilde{\alpha}_1}\right)\left(\underline{\Upsilon}^{\hbar_2}_{\tilde{\alpha}_2}\right)\right]\\ ,\left[\sqrt{\begin{matrix}1-\left(1-\overline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}+1-\left(1-\overline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}\\-(1-\left(1-\overline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1})(1-\left(1-\overline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2})\end{matrix}},\left(\overline{\Upsilon}^{\hbar_1}_{\tilde{\alpha}_1}\right)\left(\overline{\Upsilon}^{\hbar_2}_{\tilde{\alpha}_2}\right)\right]\end{pmatrix}$$

$$=\begin{pmatrix}\left[\hbar_1p_1\hbar_2p_2,\hbar_1q_1\hbar_2q_2,\hbar_1r_1\hbar_2r_2,\hbar_1s_1\hbar_2s_2\right];\\ \left[\sqrt{1-\left(1-\underline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}}),\left(\underline{\Upsilon}^{\hbar_1}_{\tilde{\alpha}_1}\right)\left(\underline{\Upsilon}^{\hbar_2}_{\tilde{\alpha}_2}\right)\right]\\ ,\left[\sqrt{1-\left(1-\overline{\Psi}^2_{\tilde{\alpha}_1}\right)^{\hbar_1}\left(1-\overline{\Psi}^2_{\tilde{\alpha}_2}\right)^{\hbar_2}}),\left(\overline{\Upsilon}^{\hbar_1}_{\tilde{\alpha}_1}\right)\left(\overline{\Upsilon}^{\hbar_2}_{\tilde{\alpha}_2}\right)\right]\end{pmatrix}.$$

If Eq. (7) holds for $\Phi=k$, that is

$$IPTFWA_w\left(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_k\right)=\begin{pmatrix}\left[\sum_{\pounds=1}^{k}\hbar_{\pounds}p_{\pounds},\sum_{\pounds=1}^{k}\hbar_{\pounds}q_{\pounds},\sum_{\pounds=1}^{k}\hbar_{\pounds}r_{\pounds},\sum_{\pounds=1}^{k}\hbar_{\pounds}s_{\pounds}\right];\\ \left[\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\underline{\Psi}^2_{\tilde{\alpha}_{\pounds}}\right)^{\hbar_{\pounds}}},\sqrt{1-\prod_{\pounds=1}^{\Phi}\left(1-\overline{\Psi}^2_{\tilde{\alpha}_{\pounds}}\right)^{\hbar_{\pounds}}}\right]\\ ,\left[\prod_{\pounds=1}^{\Phi}\underline{\Upsilon}_{\pounds}^{\hbar_{\pounds}},\prod_{\pounds=1}^{\Phi}\overline{\Upsilon}_{\pounds}^{\hbar_{\pounds}}\right]\end{pmatrix}$$

Then, if $\Phi=k+1$, by operational laws in Definition 7, we have

$$\tilde{p}_1 = ([0.3, 0.4, 0.5, 0.6]; [0.7, 0.4], [0.8, 0.3]),$$
$$\tilde{p}_2 = ([0.4, 0.5, 0.6, 0.4]; [0.9, 0.2], [0.8, 0.6]),$$
$$\tilde{p}_3 = ([0.5, 0.4, 0.6, 0.9]; [0.7, 0.6], [0.6, 0.4]),$$
$$\tilde{p}_4 = ([0.4, 0.3, 0.2, 0.1]; [0.5, 0.6], [0.7, 0.5]),$$
$$\tilde{p}_5 = ([0.5, 0.6, 0.4, 0.3]; [0.8, 0.3], [0.6, 0.5])$$

$$IPTFWA_w(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_{k+1}) = \left(\begin{array}{c} \left[\sum_{\pounds=1}^{k+1} \hbar_\pounds p_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds q_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds r_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds s_\pounds \right]; \\ \left[\sqrt{\begin{array}{c} 1 - \prod_{\pounds=1}^{k} \left(1 - \underline{\Psi}_{\tilde{\alpha}_1}^2\right)^{\hbar_\pounds} + 1 - \left(1 - \underline{\Psi}_{\tilde{\alpha}_{k+1}}^2\right)^{\hbar_{k+1}} - \\ \left(1 - \prod_{\pounds=1}^{k} \left(1 - \underline{\Psi}_{\tilde{\alpha}_\pounds}^2\right)^{\hbar_j} (1 - \left(1 - \underline{\Psi}_{\tilde{\alpha}_{k+1}}^2\right)^{\hbar_{k+1}}\right) \end{array}}, \prod_{\pounds=1}^{k+1} \underline{\Upsilon}_{\tilde{\alpha}_\pounds}^{\hbar_\pounds} \right], \\ \left[\sqrt{\begin{array}{c} 1 - \prod_{\pounds=1}^{k} \left(1 - \overline{\Psi}_{\tilde{\alpha}_1}^2\right)^{\hbar_\pounds} + 1 - \left(1 - \overline{\Psi}_{\tilde{\alpha}_{k+1}}^2\right)^{\hbar_{k+1}} - \\ \left(1 - \prod_{\pounds=1}^{k} \left(1 - \overline{\Psi}_{\tilde{\alpha}_\pounds}^2\right)^{\hbar_j} (1 - \left(1 - \overline{\Psi}_{\tilde{\alpha}_{k+1}}^2\right)^{\hbar_{k+1}}\right) \end{array}}, \prod_{\pounds=1}^{k+1} \overline{\Upsilon}_{\tilde{\alpha}_\pounds}^{\hbar_\pounds} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sum_{\pounds=1}^{k+1} \hbar_\pounds p_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds q_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds r_\pounds, \sum_{\pounds=1}^{k+1} \hbar_\pounds s_\pounds \right]; \\ , \left[\sqrt{1 - \prod_{\pounds=1}^{k+1} \left(1 - \underline{\Psi}_{\tilde{\alpha}_\pounds}^2\right)^{\hbar_\pounds}}, \prod_{\pounds=1}^{k+1} \underline{\Upsilon}_{\tilde{\alpha}_\pounds}^{\hbar_\pounds} \right], \left[\sqrt{1 - \prod_{\pounds=1}^{k+1} \left(1 - \overline{\Psi}_{\tilde{\alpha}_\pounds}^2\right)^{\hbar_\pounds}}, \prod_{\pounds=1}^{k+1} \overline{\Upsilon}_{\tilde{\alpha}_\pounds}^{\hbar_\pounds} \right] \end{array} \right).$$

Therefore Eq. (7) hold for $\Phi = k+1$. Hence Eq. (7) hold $\forall\ \Phi$. To study some properties of *(IPTFWA)* of operator, we have following Theorem.

Example 3.3: Let

be five interval Pythagorean trapezoidal fuzzy numbers, and let $\hbar = (0.10, 0.20, 0.30, 0.15, 0.25)$ be weighted vector of $\tilde{p}_\pounds\ (\pounds = 1, 2, 3, 4, 5)$. By using Eq. (7), such that

$$\left(\begin{array}{c} IPTFWA(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5) = \\ \left[\begin{array}{c} 0.10(0.3) + 0.20(0.4) + 0.30(0.5) + 0.15(0.4) + 0.25(0.5), \\ 0.10(0.4) + 0.20(0.5) + 0.30(0.4) + 0.15(0.3) + 0.25(0.6), \\ 0.10(0.5) + 0.20(0.6) + 0.30(0.6) + 0.15(0.2) + 0.25(0.4), \\ 0.10(0.6) + 0.20(0.4) + 0.30(0.9) + 0.15(0.1) + 0.25(0.3) \end{array} \right]; \\ \left[\begin{array}{c} \sqrt{1 - \left(1 - 0.7^2\right)^{.10} \left(1 - 0.9^2\right)^{.20} \left(1 - 0.7^2\right)^{.30} \left(1 - 0.5^2\right)^{.15} \left(1 - 0.8^2\right)^{.25}}, \\ \sqrt{1 - \left(1 - 0.4^2\right)^{.10} \left(1 - 0.2^2\right)^{.20} \left(1 - 0.6^2\right)^{.30} \left(1 - 0.6^2\right)^{.15} \left(1 - 0.3^2\right)^{.25}} \end{array} \right], \\ \left[(0.8)^{.10} (0.8)^{.20} (0.6)^{.30} (0.7)^{.15} (0.6)^{.25}, (0.3)^{.10} (0.6)^{.20} (0.4)^{.30} (0.5)^{.15} (0.5)^{.25} \right] \end{array} \right)$$

After simplify the above result we have interval Pythagorean trapezoidal fuzzy numbers such that $([0.44, 0.45, 0.48, 0.5]; [0.77, 0.48], [0.66, 0.46])$.

Theorem 3.4: Let $\tilde{p}_\pounds\ (j = \pounds = 1, 2, ..., \Phi)$ be a collection of *IPTF* numbers, and $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$ be the weight vector of $\tilde{p}_\pounds\ (\pounds = 1, 2, ..., \Phi)$, with $\hbar_\pounds \in [0,1]$ and $\sum_{\pounds=1}^{\Phi} \hbar_\pounds = 1$. Then, we have following properties; (1) (Idempotent): If all $\tilde{p}_\pounds\ (\pounds = 1, 2, ..., \Phi)$ are equal,

$$IPTFWA_w(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi) = \tilde{p}. \qquad (8)$$

such that $\tilde{p}_\pounds = \tilde{p}\ \forall \pounds$, then

(2) (Bounded): $\tilde{p}^- \leq IPTFWA(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi) \leq \tilde{p}^+$ for all where $\tilde{p}^- = \min_\pounds(\tilde{p}_\pounds)$ and $\tilde{p}^+ = \max_\pounds(\tilde{p}_\pounds)$.

(3) (Monotonicity): Let $\tilde{p}_\pounds^*\ (\pounds = 1, 2, ..., \Phi)$ be a collection of $I-PTF$ numbers. If $\tilde{p}_v \leq \tilde{p}_\pounds^*\ \forall\ \pounds$.

Then,

$$IPTFWA_w(\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_\Phi) \leq IPTFWA_w(\tilde{p}_1^*, \tilde{p}_2^*, ..., \tilde{p}_\Phi^*)\ \forall\ w. \quad (9)$$

Definition 3.5: Consider $\tilde{p}_\pounds\ (j = \pounds = 1, 2, ..., \Phi)$ is the group of $I-IPTF$ numbers. An induced interval Pythagorean trapezoidal fuzzy ordered weighted averaging $(I-IPTFOWA)$ operator of measurement n is a mapping and let $I-IPTFOWA$: $\Omega^\Phi -> \Omega$ having weight vector $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$ such that $\hbar_\pounds \in [0,1]$ and $\Sigma_{\pounds=1}^{\Phi} \hbar_\pounds = 1$.

$$\begin{aligned} &I-IPTFOWA(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_\Phi, \tilde{p}_\Phi \rangle) \\ &= \hbar_1 \tilde{p}_{\sigma(1)} \oplus \hbar_2 \tilde{p}_{\sigma(2)} ... \oplus \hbar_\Phi \tilde{p}_{\sigma(\Phi)} \end{aligned} \qquad (10)$$

Where $(\sigma(1), \sigma(2), ..., \sigma(\Phi))$ is a permutation of $(1, 2, ..., \Phi)$ such that $\tilde{p}_{\sigma(\pounds-1)} \geq \tilde{p}_{\sigma(\pounds)}$ for all $\pounds$, if $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$, then $I-IPTFOWA$ operator is reduced to be $I-IPTFA$ operator of dimension Φ.

Theorem 3.6: Let $\tilde{p}_\pounds\ (\pounds = 1, 2, ..., \Phi)$ be a collection of $I-IPTF$ numbers, at that point their collected an incentive by utilizing the $I-IPTFOWA$ administrator is likewise $I-IPTF$ number and

$$I-IPTFOWA(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_\Phi, \tilde{p}_\Phi \rangle)$$

$$= \left(\begin{array}{c} \left[\sum_{\pounds=1}^{\Phi} \hbar_\pounds p_{\sigma(\pounds)}, \sum_{\pounds=1}^{\Phi} \hbar_\pounds q_{\sigma(\pounds)}, \sum_{\pounds=1}^{\Phi} \hbar_\pounds r_{\sigma(\pounds)}, \sum_{\pounds=1}^{\Phi} \hbar_\pounds s_{\sigma(\pounds)} \right]; \\ \left[\sqrt{1 - \prod_{\pounds=1}^{\Phi} \left(1 - \underline{\Psi}_{\sigma(\pounds)}^2\right)^{\hbar_\pounds}}, \sqrt{1 - \prod_{\pounds=1}^{\Phi} \left(1 - \overline{\Psi}_{\sigma(\pounds)}^2\right)^{\hbar_\pounds}} \right], \\ \left[\prod_{\pounds=1}^{\Phi} \underline{\Upsilon}_{\sigma(\pounds)}^{\hbar_\pounds}, \prod_{\pounds=1}^{\Phi} \overline{\Upsilon}_{\sigma(\pounds)}^{\hbar_\pounds} \right] \end{array} \right) \qquad (11)$$

where $\hbar = \left(\frac{1}{\Phi}, \frac{1}{\Phi}, ..., \frac{1}{\Phi}\right)^T$ having weight of $\tilde{p}_{£}\left(j = £ = 1, 2..., \Phi\right)$ with $\hbar_{£} \in [0,1]$ and $\sum_{£=1}^{\Phi} \hbar_{£} = 1$.

Example 3.7: *Let*

$$\langle u_1, \tilde{p}_1 \rangle = \langle 0.3, ([0.3, 0.4, 0.5, 0.6]; [0.8, 0.4], [0.6, 0.3]) \rangle,$$
$$\langle u_2, \tilde{p}_2 \rangle = \langle 0.2, ([0.4, 0.5, 0.6, 0.4]; [0.9, 0.4], [0.8, 0.3]) \rangle,$$
$$\langle u_3, \tilde{p}_3 \rangle = \langle 0.5, ([0.6, 0.5, 0.4, 0.8]; [0.7, 0.6], [0.6, 0.4]) \rangle,$$
$$\langle u_4, \tilde{p}_4 \rangle = \langle 0.1, ([0.4, 0.3, 0.2, 0.1]; [0.6, 0.6], [0.6, 0.5]) \rangle,$$
$$\langle u_5, \tilde{p}_5 \rangle = \langle 0.7, ([0.6, 0.4, 0.2, 0.1]; [0.8, 0.5], [0.6, 0.5]) \rangle.$$

be five induced interval Pythagorean trapezoidal fuzzy numbers, and let $\hbar = (0.15, 0.25, 0.10, 0.20, 0.30)$ be weighted vector of $\tilde{p}_{£}\left(£ = 1, 2, 3, 4, 5\right)$. We write an ordering form with the help of first component such that

$$\langle u_5, \tilde{p}_5 \rangle = \langle 0.7, ([0.6, 0.4, 0.2, 0.1]; [0.8, 0.5], [0.6, 0.5]) \rangle$$
$$\langle u_3, \tilde{p}_3 \rangle = \langle 0.5, ([0.6, 0.5, 0.4, 0.8]; [0.7, 0.6], [0.6, 0.4]) \rangle,$$
$$\langle u_1, \tilde{p}_1 \rangle = \langle 0.3, ([0.3, 0.4, 0.5, 0.6]; [0.8, 0.4], [0.6, 0.3]) \rangle,$$
$$\langle u_2, \tilde{p}_2 \rangle = \langle 0.2, ([0.4, 0.5, 0.6, 0.4]; [0.9, 0.4], [0.8, 0.3]) \rangle,$$
$$\langle u_4, \tilde{p}_4 \rangle = \langle 0.1, ([0.4, 0.3, 0.2, 0.1]; [0.6, 0.6], [0.6, 0.5]) \rangle$$

The ordering includes the ordered Pythagorean trapezoidal fuzzy arguments.

$$\tilde{p}_{\sigma(1)} = \langle 0.7, ([0.6, 0.4, 0.2, 0.1]; [0.8, 0.5], [0.6, 0.5]) \rangle$$
$$\tilde{p}_{\sigma(2)} = \langle 0.5, ([0.6, 0.5, 0.4, 0.8]; [0.7, 0.6], [0.6, 0.4]) \rangle,$$
$$\tilde{p}_{\sigma(3)} = \langle 0.3, ([0.3, 0.4, 0.5, 0.6]; [0.8, 0.4], [0.6, 0.3]) \rangle,$$
$$\tilde{p}_{\sigma(4)} = \langle 0.2, ([0.4, 0.5, 0.6, 0.4]; [0.9, 0.4], [0.8, 0.3]) \rangle,$$
$$\tilde{p}_{\sigma(5)} = \langle 0.1, ([0.4, 0.3, 0.2, 0.1]; [0.6, 0.6], [0.6, 0.5]) \rangle$$

By using Eq. (11) we have,

$$\left(\begin{array}{c} I - IPTFOWA\left(\langle u_1, \tilde{p}_1 \rangle \langle u_2, \tilde{p}_2 \rangle, \langle u_3, \tilde{p}_3 \rangle, \langle u_4, \tilde{p}_4 \rangle, \langle u_5, \tilde{p}_5 \rangle, \right) = \\ \begin{bmatrix} 0.15(0.6) + 0.25(0.6) + 0.10(0.3) + 0.20(0.4) + 0.30(0.4), \\ 0.15(0.4) + 0.25(0.5) + 0.10(0.4) + 0.20(0.5) + 0.30(0.3), \\ 0.15(0.2) + 0.25(0.4) + 0.10(0.5) + 0.20(0.6) + 0.30(0.2), \\ 0.15(0.1) + 0.25(0.8) + 0.10(0.6) + 0.20(0.4) + 0.30(0.1), \end{bmatrix}; \\ \begin{bmatrix} \sqrt{1 - \left(1 - 0.8^2\right)^{.15} \left(1 - 0.7^2\right)^{.25} \left(1 - 0.8^2\right)^{.10} \left(1 - 0.9^2\right)^{.20} \left(1 - 0.6^2\right)^{.30}}, \\ \sqrt{1 - \left(1 - 0.5^2\right)^{.15} \left(1 - 0.6^2\right)^{.25} \left(1 - 0.4^2\right)^{.10} \left(1 - 0.4^2\right)^{.20} \left(1 - 0.6^2\right)^{.30}} \end{bmatrix}, \\ \left[(0.6)^{.15} (0.6)^{.25} (0.6)^{.10} (0.8)^{.20} (0.6)^{.30}, (0.5)^{.15} (0.4)^{.25} (0.3)^{.10} (0.3)^{.20} (0.5)^{.30} \right] \end{array} \right)$$

After simplify the above result we have induced interval Pythagorean trapezoidal fuzzy numbers such that $([0.39, 0.41, 0.36, 0.32]; [0.77, 0.55], [0.63, 0.40])$.

Theorem 3.8: Consider $\tilde{p}_{£}\left(j = £ = 1, 2, ..., \Phi\right)$ is a group of $I - IPTF$ numbers, and $\hbar = \left(\hbar_1, \hbar_2, ..., \hbar_{\Phi}\right)^T$ is the weight vector of $\tilde{p}_{£} = \left(£ = 1, 2, ..., \Phi\right)$, with $\hbar_{£} \in [0,1]$ and $\sum_{£=1}^{\Phi} \hbar_{£} = 1$. Then, we have following.

(1) (Idempotent): If all $\tilde{p}_{£}\left(j = £ = 1, 2, ..., \Phi\right)$ are equal, such that, $\tilde{p}_{£} = \tilde{p} \ \forall \ £$, then

$$I - IPTFOWA_w\left(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_{\Phi}, \tilde{p}_{\Phi} \rangle\right) = \tilde{p}. \quad (12)$$

(2) (Boundary):

$$\tilde{p}^- \le I - IPTFOWA\left(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_{\Phi}, \tilde{p}_{\Phi} \rangle\right) \le \tilde{p}^+$$

for all $\hbar$, where $\tilde{p}^- = \min_{£}\left(\tilde{p}_{£}\right)$ and $\tilde{p}^+ = \max_{£}\left(\tilde{p}_{£}\right)$.

(3) (Monotonicity):

Definition 3.9: Consider $\tilde{p}_{£}^*\left(£ = 1, 2, ..., \Phi\right)$ is the group of $I - IPTF$ numbers. If $\tilde{p}_{£} \le \tilde{p}_{£}^* \forall \ £$, then

$$I - IPTFOWA_w\left(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_{\Phi}, \tilde{p}_{\Phi} \rangle\right)$$
$$\le I - IPTFOWA_w\left(\left\langle u_1, \tilde{p}_1^* \right\rangle, \left\langle u_2, \tilde{p}_2^* \right\rangle, ..., \left\langle u_{\Phi}, \tilde{p}_{\Phi}^* \right\rangle\right) \ \forall \ \hbar. \quad (13)$$

Theorem 3.10: Let $\tilde{p}_{£}\left(j = £ = 1, 2, ..., \Phi\right)$ be a gathering of $I - IPTF$ numbers, and $\hbar = \left(\hbar_1, \hbar_2, ..., \hbar_{\Phi}\right)^T$ be the weight vector of $I - IPTFOWA$ operator, with $\hbar_{£} \in [0,1]$ and $\sum_{£=1}^{\Phi} \hbar_{£} = 1$, hence we have

(1) If $\hbar = (1, 0, ..., 0)^T$, then

$$I - IPTFOWA_w\left(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_{\Phi}, \tilde{p}_{\Phi} \rangle\right) = \max_{£}\left(\tilde{p}_{£}\right).$$

(2) If $\hbar = (0, 0, ..., 1)^T$, then

$$I - IPTFOWA_w\left(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_{\Phi}, \tilde{p}_{\Phi} \rangle\right) = \min_{£}\left(\tilde{p}_{£}\right).$$

(3) If $\hbar = 1, \ w_i = 0,$ and $i \ne £$ then

$$I - IPTFOWA_w\left(\langle u_1, \tilde{p}_1 \rangle, \langle u_2, \tilde{p}_2 \rangle, ..., \langle u_{\Phi}, \tilde{p}_{\Phi} \rangle\right) = \tilde{p}_{\sigma(£)},$$

Where $\tilde{p}_{\sigma(£)}$ is the jth largest of $\tilde{p}_i\left(i = 1, 2, ..., \Phi\right)$.

We shall define Induced interval Pythagorean trapezoidal fuzzy hybrid averaging $\left(I - IPTFHA\right)$ operator in the resulting:

Theorem 3.11: Let $\tilde{p}_{£}\left(j=£=1,2,...,\Phi\right)$ be a collection of $I-IPTF$ numbers. An induced Interval Pythagorean trapezoidal fuzzy hybrid averaging $(I-IPTFHA)$ administrator of measurement n is a mapping $(I-IPTFHA):\ \Omega^{\Phi}->\Omega$ and $\hbar=(\hbar_1,\hbar_2,...,\hbar_{\Phi})^T$ having weight

$\tilde{p}_{£}\left(j=£=1,2,...,\Phi\right)$, with $\hbar_{£}\in[0,1]$ and $\Sigma_{£=1}^{\Phi}\hbar_{£}=1$.

$$I-IPTFHA_{w,w}\left(\langle u_1,\tilde{p}_1\rangle,\langle u_2,\tilde{p}_2\rangle,...,\langle u_{\Phi},\tilde{p}_{\Phi}\rangle\right)$$
$$=\hbar_1\dot{\tilde{p}}_{\sigma(1)}\oplus\hbar_1\dot{\tilde{p}}_{\sigma(2)}...\oplus\hbar_1\dot{\tilde{p}}_{\sigma(\Phi)} \quad (14)$$

where $\dot{\tilde{p}}_{\sigma(£)}$ is the $£th$ largest of the weighted

$I-PTF$ numbers $\dot{\tilde{p}}_{£}\left(\dot{\tilde{p}}_{£}=\Phi\hbar_{£}\dot{\tilde{p}}_{£}\right)$, and

$\hbar=(\hbar_1,\hbar_2,...,\hbar_{\Phi})^T$ be the weight vector of $\tilde{p}_{£}$ operator, with $\hbar_{£}\in[0,1]$ and $\Sigma_{£=1}^{\Phi}\hbar_{£}=1$. Where n is the balancing coefficient, which demonstrates a character of unfaltering quality in such a case, if the vector

$\hbar=(\hbar_1,\hbar_2,...,\hbar_{\Phi})^T$ approaches $\left(\frac{1}{\Phi},\frac{1}{\Phi},...,\frac{1}{\Phi}\right)^T$, then the vector $(\Phi w_1\tilde{p}_1,\Phi w_2\tilde{p}_2,...,\Phi w_{\Phi}\tilde{p}_{\Phi})^T$ approaches $(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_{\Phi})^T$.

Theorem 3.12: Let $\tilde{p}_{£}\left(j=£=1,2,...,\Phi\right)$ be a collection of $I-PTF$ numbers, then their aggregated value by using the $(I-PTFHA)$ operator is also $I-PTF$ number and

$$I-PTFHA_{w,w}\left(\langle u_1,\tilde{p}_1\rangle,\langle u_2,\tilde{p}_2\rangle,...,\langle u_{\Phi},\tilde{p}_{\Phi}\rangle\right)$$
$$=\begin{pmatrix}\left[\sum_{£=1}^{\Phi}\hbar_{£}\dot{\tilde{p}}_{\sigma(£)},\sum_{£=1}^{\Phi}\hbar_{£}\dot{\tilde{q}}_{\sigma(£)},\sum_{£=1}^{\Phi}\hbar_{£}\dot{\tilde{r}}_{\sigma(£)},\sum_{£=1}^{\Phi}\hbar_{£}\dot{\tilde{s}}_{\sigma(£)}\right]; \\ \left[\sqrt{1-\prod_{£=1}^{\Phi}\left(1-\dot{\underline{\Psi}}^2_{\sigma(£)}\right)^{\hbar_{£}}},\sqrt{1-\prod_{£=1}^{\Phi}\left(1-\dot{\bar{\Psi}}^2_{\sigma(£)}\right)^{\hbar_{£}}}\right], \\ \left[\prod_{£=1}^{\Phi}\dot{\underline{\Upsilon}}_{\sigma(£)}^{\hbar_{£}},\prod_{£=1}^{\Phi}\dot{\bar{\Upsilon}}_{\sigma(£)}^{\hbar_{£}}\right]\end{pmatrix}. \quad (15)$$

Theorem 3.13: The $IPTFWA$ administrator is an alternate instance of the $I-IPTFHA$ administrator.

Proof: Let $\hbar=\left(\frac{1}{\Phi},\frac{1}{\Phi},...,\frac{1}{\Phi}\right)^T$, then

$$I-IPTFHA_{w,w}\left(\langle u_1,\tilde{p}_1\rangle,\langle u_2,\tilde{p}_2\rangle,...,\langle u_{\Phi},\tilde{p}_{\Phi}\rangle\right)$$
$$=\left(\hbar_1\dot{\tilde{p}}_{\sigma(1)}\oplus\hbar_2\dot{\tilde{p}}_{\sigma(2)}...\oplus\hbar_{\Phi}\dot{\tilde{p}}_{\sigma(\Phi)}\right)$$
$$=\left(\frac{1}{\Phi}\dot{\tilde{p}}_{\sigma(1)}\oplus\frac{1}{\Phi}\dot{\tilde{p}}_{\sigma(2)}...\oplus\frac{1}{\Phi}\dot{\tilde{p}}_{\sigma(\Phi)}\right)$$
$$=(\hbar_1\tilde{p}\oplus\hbar_2\tilde{p}...\oplus\hbar_{\Phi}\tilde{p})$$
$$=IPTFWA_w(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_{\Phi}).$$

Theorem 3.14: The $I-IPTFOWA$ operator is a different case of the $I-IPTFHA$ operator.

Proof: Let $\hbar=\left(\frac{1}{\Phi},\frac{1}{\Phi},...,\frac{1}{\Phi}\right)^T$, then

$$\left(\dot{\tilde{p}}_{£}=\tilde{p}_{£},£=1,2,...,\Phi\right), \text{ hence}$$

$$I-IPTFHA_{w,w}\left(\langle u_1,\tilde{p}_1\rangle,\langle u_2,\tilde{p}_2\rangle,...,\langle u_{\Phi},\tilde{p}_{\Phi}\rangle\right)$$
$$=\left(\hbar_1\dot{\tilde{p}}_{\sigma(1)}\oplus\hbar_2\dot{\tilde{p}}_{\sigma(2)}...\oplus\hbar_{\Phi}\dot{\tilde{p}}_{\sigma(\Phi)}\right)$$
$$=\left(\hbar_1\tilde{p}_{\sigma(1)}\oplus\hbar_2\tilde{p}_{\sigma(2)}...\oplus\hbar_{\Phi}\tilde{p}_{\sigma(\Phi)}\right)$$
$$=I-IPTFOWA_w$$
$$=\left(\langle u_1,\tilde{p}_1\rangle,\langle u_2,\tilde{p}_2\rangle,...,\langle u_{\Phi},\tilde{p}_{\Phi}\rangle\right).$$

4. An Application of Interval Pythagorean Trapezoidal Fuzzy Numbers with MAGDM Problems

To solve the *MAGDM* problem we use $I-IPTFOWA$ as well as $I-IPTFHA$ operators with interval Pythagorean trapezoidal fuzzy information.

Let $B=\{B_1,B_2,...,B_m\}$ be set a of alternatives and the weighting vector of the attribute $C=\{C_1,C_2,...,C_n\}$ is, $P=\{P_1,P_2,...,P_n\}$, where sum of $p_{£}$ is equal to one such that $\Sigma_{£=1}^{\Phi}P_{£}=1$. Let the set of decision makers is denoted by $Q=\{Q_1,Q_2,...,Q_t\}$ whose weighting vector is $\hbar=(\hbar_1,\hbar_2,...,\hbar_{\Phi})^T$ such that, $\hbar_k\in[0,1]$ and $\Sigma_{k=1}^{t}\hbar_k=1$. Consider that

$$\tilde{Z}^k=\left(\tilde{z}_{i£}^k\right)_{m\times n}=\left[p_{i£}^k,q_{i£}^k,r_{i£}^k,s_{i£}^k\right];[\Psi_{i£}^k,\bar{\Psi}_{i£}^k],[\Upsilon_{i£}^k,\bar{\Upsilon}_{i£}^k])_{m\times n} \quad (16)$$

is the interval Pythagorean trapezoidal fuzzy decision matrix,

$[\Psi_{i£}^k,\bar{\Psi}_{i£}^k]\subset[0,1]$, and $[\Upsilon_{i£}^k,\bar{\Upsilon}_{i£}^k]\subset[0,1]$ $\Psi_{i£}^{2k}+\bar{\Psi}_{i£}^{2k}\le 1$ and $\Upsilon_{i£}^{2k}+\bar{\Upsilon}_{i£}^{2k}\le 1$ $(£=1,2,...,\Phi,i=1,2,...,m,k=1,2,...,t)$.

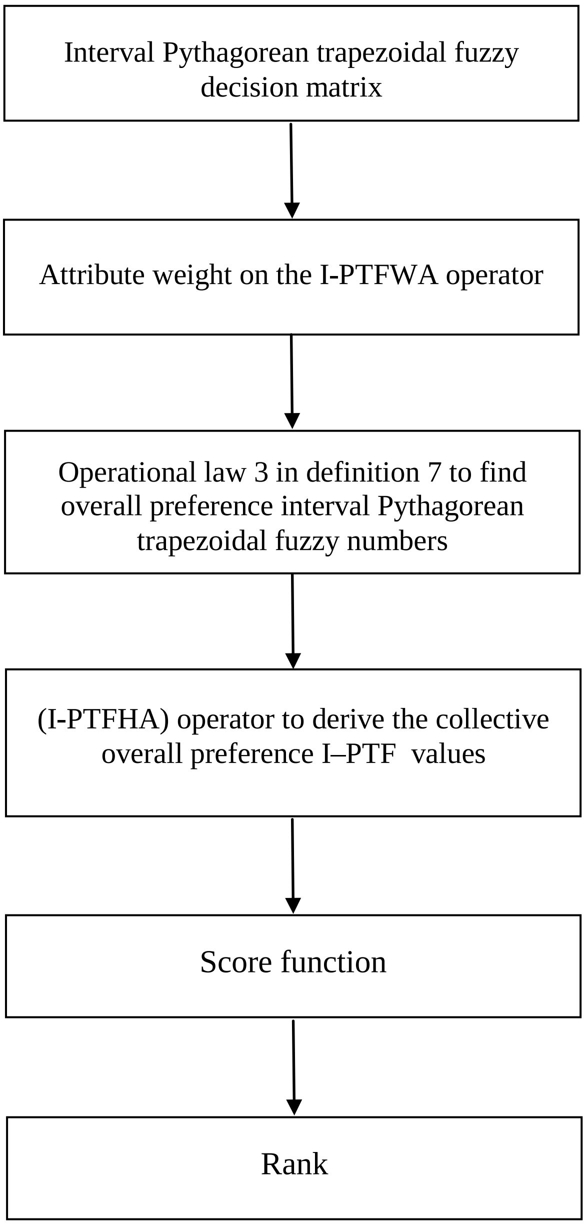

Fig. 1: Flow chart of proposed algorithm

In the following steps we solve *MAGDM* problems by applying *IPTF* information:

Step 1: In this step we construct interval Pythagorean trapezoidal fuzzy decision matrix from Table 1-4.

Step 2: In this step we construct interval Pythagorean trapezoidal fuzzy ordered decision matrix from Table 1-4.

Step 3: Aggregate all the preference values by applying $I-IPTFOWA$ operator and get overall preference values.

Step 4: Applying the known weight by using operational law 3 in definition 7 to find overall preference interval Pythagorean trapezoidal fuzzy values $\left(\tilde{z}_i^k\right)$ of the alternative B_i.

$$(\tilde{z}_i) = ([p_i, q_i, r_i, s_i]; [\Psi_i, \bar{\Psi}_i], [\Upsilon_i, \bar{\Upsilon}_i]) \\ = I - IPTFHA_{w,w}(z_{i£}^1, z_{i£}^2, ..., z_{i£}^t), \quad (17)$$

Step 5: Applying the $(I-IPTFHA)$ operator to stem the communal whole partiality $IPTF$ values $\tilde{z}_i\,(i = 1, 2, ..., m)$ of the alternative B_i; where $\hbar = (\hbar_1, \hbar_2, ..., \hbar_\Phi)^T$) be the weight vector of decision makers. With $\hbar_k \in [0,1]$ and $\Sigma_{k=1}^t \hbar_k = 1$; $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_t)^T$ is the associated weight vector of the $(I-IPTFHA)$ operator, with $\Gamma_k \in [0,1]$ and $\Sigma_{k=1}^t \Gamma_k = 1$.

Step 6: In this step we use score function to aggregate value of each alternative.

Step 7: In this step we determine the rank of alternative B_i and select the best option according to descending order.

5. Numerical Example

International ecological disquiet is a certainty, and deliberation on the black fabrication in several industries. A car company wanted to choose the most suitable black supplier having the key factor in its industrial process. Subsequently pre-evaluation, four suppliers $Bi(i = 1, 2, 3, 4)$ have persisted as alternatives for further evaluation. There are four criteria to be supposed such that: C_1 creation worth; C_2 equipment competence; C_3 contamination control; C_4 atmosphere supervision (having weight $\hbar = (0.4, 0.3, 0.2, 0.1)^T$. The company arranged four group DMS form four fidelity branches; q_1 is from the engineering branch; q_2 is from the acquiring branch; q_3 is from the quality assessment branch; q_4 is from the fabrication branch; (having weight $w = (0.20, 0.30, 0.35, 0.15)^T$. They constructed the decision matrix $Z^{(k)} = \left(z_{ij}^{(k)}\right)_{4\times 4}$

(k = 1, 2, 3, 4) as follows:

Step 1: The decision makers give his decision in the following tables.

Table 1: Decision matrix of Expert 1

	<0.2,([0.4,0.5,0.2,0.3];[0.6,0.5],[0.4,0.7])>	<0.5,([0.7,0.5,0.6,0.3];[0.4,0.6],[0.5,0.6])>
	<0.4,([0.4,0.5,0.7,0.2];[0.6,0.7],[0.7,0.6])>	<0.5,([0.3,0.1,0.2,0.4];[0.3,0.8],[0.6,0.5])>
	<0.2,([0.6,0.8,0.9,0.2];[0.3,0.9],[0.8,0.4])>	<0.6,([0.2,0.1,0.3,0.5];[0.6,0.5],[0.3,0.8])>
$Z^{(1)}=$	<0.3,([0.4,0.5,0.4,0.3];[0.5,0.5],[0.8,0.6])>	<0.5,([0.4,0.3,0.4,0.2];[0.3,0.8],[0.6,0.5])>
	<0.3,([0.3,0.4,0.5,0.6];[0.8,0.4],[0.5,0.7])>	<0.6,([0.3,0.4,0.4,0.3];[0.5,0.6],[0.4,0.7])>
	<0.7,([0.4,0.3,0.6,0.3];[0.4,0.6],[0.6,0.5])>	<0.1,([0.4,0.3,0.2,0.1];[0.4,0.6],[0.7,0.5])>
	<0.3,([0.3,0.1,0.2,0.3];[0.3,0.8],[0.8,0.6])>	<0.8,([0.5,0.3,0.6,0.4];[0.5,0.7],[0.8,0.5])>
	<0.4,([0.4,0.3,0.4,0.6];[0.4,0.7],[0.6,0.4])>	<0.9,([0.9,0.6,0.4,0.1];[0.3,0.8],[0.6,0.5])>

Table 2: Decision matrix of Expert 2

	<0.4,([0.3,0.4,0.5,0.6];[0.4,0.8],[0.8,0.5])>	<0.6,([0.4,0.3,0.6,0.7];[0.4,0.6],[0.6,0.6])>
	<0.9,([0.3,0.4,0.4,0.6];[0.3,0.7],[0.8,0.3])>	<0.2,([0.4,0.5,0.3,0.1];[0.3,0.9],[0.8,0.3])>
	<0.5,([0.3,0.4,0.5,0.6];[0.5,0.5],[0.6,0.7])>	<0.6,([0.4,0.5,0.7,0.8];[0.6,0.7],[0.4,0.7])>
$Z^{(2)}=$	<0.2,([0.4,0.3,0.1,0.3];[0.8,0.6],[0.4,0.6])>	<0.5,([0.3,0.4,0.5,0.7];[0.7,0.6],[0.5,0.8])>
	<0.3,([0.8,0.2,0.3,0.4];[0.5,0.5],[0.6,0.7])>	<0.7,([0.4,0.5,0.7,0.5];[0.3,0.9],[0.8,0.3])>
	<0.4,([0.4,0.3,0.2,0.1];[0.6,0.5],[0.5,0.6])>	<0.5,([0.3,0.4,0.5,0.6];[0.9,0.3],[0.3,0.8])>
	<0.8,([0.5,0.4,0.2,0.3];[0.4,0.7],[0.8,0.5])>	<0.3,([0.5,0.5,0.4,0.6];[0.3,0.8],[0.6,0.4])>
	<0.4,([0.4,0.5,0.6,0.7];[0.4,0.8],[0.7,0.3])>	<0.7,([0.4,0.6,0.3,0.4];[0.8,0.6],[0.5,0.7])>

Table 3: Decision matrix of Expert 3

	<0.1,([0.4,0.5,0.6,0.9];[0.2,0.8],[0.7,0.4])>	<0.5,([0.3,0.4,0.5,0.4];[0.3,0.7],[0.7,0.6])>
	<0.9,([0.9,0.6,0.3,0.4];[0.4,0.8],[0.6,0.4])>	<0.4,([0.4,0.5,0.6,0.7];[0.5,0.8],[0.8,0.4])>
	<0.1,([0.4,0.5,0.6,0.5];[0.6,0.5],[0.7,0.8])>	<0.5,([0.7,0.6,0.3,0.2];[0.8,0.2],[0.5,0.8])>
$Z^{(3)}=$	<0.8,([0.4,0.5,0.1,0.2];[0.4,0.8],[0.8,0.6])>	<0.5,([0.8,0.4,0.3,04];[0.4,0.8],[0.7,0.5])>
	<0.3,([0.6,0.2,0.3,0.4];[0.5,0.6],[0.6,0.5])>	<0.6,([0.4,0.5,0.6,0.5];[0.4,0.8],[0.6,0.4])>
	<0.7,([0.4,0.5,0.4,0.1];[0.3,0.7],[0.8,0.4])>	<0.2,([0.3,0.4,0.5,0.9];[0.5,0.6],[0.8,0.5])>
	<0.2,([0.4,0.6,0.2,0.3];[0.5,0.7],[0.8,0.4])>	<0.1,([0.6,0.7,0.8,0.6];[0.4,0.9],[0.8,0.3])>
	<0.2,([0.4,0.6,0.3,0.4];[0.4,0.8],[0.8,0.5])>	<0.9,([0.3,0.4,0.3,0.5];[0.4,0.7],[0.8,0.3])>

Table 4: Decision matrix of Expert 4

	<0.2,([0.1,0.4,0.5,0.3];[0.4,0.8],[0.8,0.5])>	<0.3,([0.3,0.4,0.5,0.6];[0.5,0.8],[0.7,0.4])>
	<0.5,([0.4,0.2,0.2,0.5];[0.3,0.8],[0.8,0.4])>	<0.2,([0.4,0.6,0.3,0.2];[0.8,0.3],[0.2,0.9])>
	<0.9,([0.3,0.4,0.5,0.7];[0.4,0.7],[0.7,0.4])>	<0.3,([0.6,0.8,0.3,0.2];[0.3,0.8],[0.8,0.4])>
$Z^{(4)}=$	<0.3,([0.3,0.5,0.6,0.9];[0.4,0.7],[0.8,0.5])>	<0.6,([0.6,0.3,0.1,0.4];[0.5,0.6],[0.7,0.5])>
	<0.6,([0.9,0.3,0.1,0.2];[0.5,0.6],[0.8,0.4])>	<0.1,([0.6,0.2,0.5,0.1];[0.5,0.7],[0.7,0.6])>
	<0.3,([0.4,0.7,0.8,0.9];[0.4,0.7],[0.5,0.6])>	<0.4,([0.4,0.5,0.6,0.4];[0.6,0.5],[0.8,0.8])>
	<0.7,([0.4,0.6,0.3,0.5];[0.7,0.8],[0.6,0.5])>	<0.1,([0.5,0.8,0.7,0.3];[0.4,0.7],[0.8,0.3])>
	<0.3,([0.3,0.4,0.7,0.2];[0.7,0.5],[0.5,0.8])>	<0.8,([0.4,0.9,0.6,0.3];[0.5,0.5],[0.6,0.7])>

Step 2: Pythagorean trapezoidal fuzzy ordered decision matrix

Table 1: Decision matrix of Expert 1

	<0.6,([0.3,0.4,0.4,0.3];[0.5,0.6],[0.4,0.7])>	<0.5,([0.7,0.5,0.6,0.3];[0.4,0.6],[0.5,0.6])>
	<0.7,([0.4,0.3,0.6,0.3];[0.4,0.6],[0.6,0.5])>	<0.5,([0.4,0.3,0.4,0.2];[0.3,0.8],[0.6,0.5])>
	<0.8,([0.5,0.3,0.6,0.4];[0.5,0.7],[0.8,0.5])>	<0.6,([0.2,0.1,0.3,0.5];[0.6,0.5],[0.3,0.8])>
$Z^{(1}=$	<0.9,([0.9,0.6,0.4,0.1];[0.3,0.8],[0.6,0.5])>	<0.5,([0.4,0.3,0.4,0.2];[0.3,0.8],[0.6,0.5])>
	<0.3,([0.3,0.4,0.5,0.6];[0.8,0.4],[0.5,0.7])>	<0.2,([0.4,0.5,0.2,0.3];[0.6,0.5],[0.4,0.7])>
	<0.4,([0.4,0.5,0.7,0.2];[0.6,0.7],[0.7,0.6])>	<0.1,([0.4,0.3,0.2,0.1];[0.4,0.6],[0.7,0.5])>
	<0.3,([0.4,0.5,0.4,0.3];[0.5,0.5],[0.8,0.6])>	<0.2,([0.6,0.8,0.9,0.2];[0.3,0.9],[0.8,0.4])>
	<0.4,([0.4,0.3,0.4,0.6];[0.4,0.7],[0.6,0.4])>	<0.3,([0.3,0.1,0.2,0.3];[0.3,0.8],[0.8,0.6])>

Table 2: Decision matrix of Expert 2

	<0.7,([0.4,0.5,0.7,0.5];[0.3,0.9],[0.8,0.3])>	<0.6,([0.4,0.3,0.6,0.7];[0.4,0.6],[0.6,0.6])>
	<0.9,([0.3,0.4,0.4,0.6];[0.3,0.7],[0.8,0.3])>	<0.5,([0.3,0.4,0.5,0.6];[0.9,0.3],[0.3,0.8])>
	<0.8,([0.5,0.4,0.2,0.3];[0.4,0.7],[0.8,0.5])>	<0.6,([0.4,0.5,0.7,0.8];[0.6,0.7],[0.4,0.7])>
$Z^{(2)}=$	<0.7,([0.4,0.6,0.3,0.4];[0.8,0.6],[0.5,0.7])>	<0.5,([0.3,0.4,0.5,0.7];[0.7,0.6],[0.5,0.8])>
	<0.3,([0.8,0.2,0.3,0.4];[0.5,0.5],[0.6,0.7])>	<0.4,([0.4,0.3,0.2,0.1];[0.6,0.5],[0.5,0.6])>
	<0.2,([0.4,0.5,0.3,0.1];[0.3,0.9],[0.8,0.3])>	<0.5,([0.3,0.4,0.5,0.6];[0.5,0.5],[0.6,0.7])>
	<0.5,([0.3,0.4,0.5,0.6];[0.5,0.5],[0.6,0.7])>	<0.3,([0.5,0.5,0.4,0.6];[0.3,0.8],[0.6,0.4])>
	<0.4,([0.4,0.5,0.6,0.7];[0.4,0.8],[0.7,0.3])>	<0.2,([0.4,0.3,0.1,0.3];[0.8,0.6],[0.4,0.6])>

Table 3: Decision matrix of Expert 3

	<0.6,([0.4,0.5,0.6,0.5];[0.4,0.8],[0.6,0.4])>	<0.5,([0.3,0.4,0.5,0.4];[0.3,0.7],[0.7,0.6])>
	<0.9,([0.9,0.6,0.3,0.4];[0.4,0.8],[0.6,0.4])>	<0.7,([0.4,0.5,0.4,0.1];[0.3,0.7],[0.8,0.4])>
	<0.8,([0.4,0.5,0.1,0.2];[0.4,0.8],[0.8,0.6])>	<0.5,([0.7,0.6,0.3,0.2];[0.8,0.2],[0.5,0.8])>
$Z^{(3)}=$	<0.9,([0.3,0.4,0.3,0.5];[0.4,0.7],[0.8,0.3])>	<0.5,([0.8,0.4,0.3,04];[0.4,0.8],[0.7,0.5])>
	<0.3,([0.6,0.2,0.3,0.4];[0.5,0.6],[0.6,0.5])>	<0.1,([0.4,0.5,0.6,0.9];[0.2,0.8],[0.7,0.4])>
	<0.4,([0.4,0.5,0.6,0.7];[0.5,0.8],[0.8,0.4])>	<0.2,([0.3,0.4,0.5,0.9];[0.5,0.6],[0.8,0.5])>
	<0.2,([0.4,0.6,0.2,0.3];[0.5,0.7],[0.8,0.4])>	<0.1,([0.6,0.7,0.8,0.6];[0.4,0.9],[0.8,0.3])>
	<0.2,([0.4,0.6,0.3,0.4];[0.4,0.8],[0.8,0.5])>	<0.1,([0.4,0.5,0.6,0.5];[0.6,0.5],[0.7,0.8])>

Table 4: Decision matrix of Expert 4

	<0.6,([0.9,0.3,0.1,0.2];[0.5,0.6],[0.8,0.4])>	<0.3,([0.3,0.4,0.5,0.6];[0.5,0.8],[0.7,0.4])>
	<0.5,([0.4,0.2,0.2,0.5];[0.3,0.8],[0.8,0.4])>	<0.4,([0.4,0.5,0.6,0.4];[0.6,0.5],[0.8,0.8])>
	<0.9,([0.3,0.4,0.5,0.7];[0.4,0.7],[0.7,0.4])>	<0.7,([0.4,0.6,0.3,0.5];[0.7,0.8],[0.6,0.5])>
$Z^{(4)}=$	<0.8,([0.4,0.9,0.6,0.3];[0.5,0.5],[0.6,0.7])>	<0.6,([0.6,0.3,0.1,0.4];[0.5,0.6],[0.7,0.5])>
	<0.2,([0.1,0.4,0.5,0.3];[0.4,0.8],[0.8,0.5])>	<0.1,([0.6,0.2,0.5,0.1];[0.5,0.7],[0.7,0.6])>
	<0.3,([0.4,0.7,0.8,0.9];[0.4,0.7],[0.5,0.6])>	<0.2,([0.4,0.6,0.3,0.2];[0.8,0.3],[0.2,0.9])>
	<0.3,([0.6,0.8,0.3,0.2];[0.3,0.8],[0.8,0.4])>	<0.1,([0.5,0.8,0.7,0.3];[0.4,0.7],[0.8,0.3])>
	<0.3,([0.3,0.4,0.7,0.2];[0.7,0.5],[0.5,0.8])>	<0.3,([0.3,0.5,0.6,0.9];[0.4,0.7],[0.8,0.5])>

Step 3: Apply the induced interval Pythagorean trapezoidal fuzzy ordered weighted averaging $(I-IPTFOWA)$ operator to aggregate all the individual Pythagorean trapezoidal fuzzy ordered decision matrix, $Z^{(k)}$ $(k=1,2,3,4)$ of the alternative B_i.

$\tilde{z}_1^{(1)} = ([0.43, 0.44, 0.46, 0.36]; [0.59, 0.57], [0.44, 0.46])$
$\tilde{z}_1^{(2)} = ([0.37, 0.31, 0.46, 0.29]; [0.43, 0.71], [0.62, 0.51])$
$\tilde{z}_1^{(3)} = ([0.38, 0.25, 0.46, 0.39]; [0.49, 0.72], [0.59, 0.58])$
$\tilde{z}_1^{(4)} = ([0.96, 0.44, 0.40, 0.25]; [0.36, 0.76], [0.61, 0.48])$

$\tilde{z}_2^{(1)} = ([0.47, 0.37, 0.57, 0.55]; [0.40, 0.78], [0.69, 0.46])$
$\tilde{z}_2^{(2)} = ([0.33, 0.39, 0.38, 0.45]; [0.69, 0.64], [0.54, 0.48])$
$\tilde{z}_2^{(3)} = ([0.43, 0.44, 0.43, 0.54]; [0.49, 0.69], [0.59, 0.57])$
$\tilde{z}_2^{(4)} = ([0.37, 0.49, 0.40, 0.54]; [0.73, 0.67], [0.52, 0.60])$

$\tilde{z}_3^{(1)} = ([0.41, 0.41, 0.51, 0.49]; [0.39, 0.74], [0.63, 0.47])$
$\tilde{z}_3^{(2)} = ([0.62, 0.53, 0.41, 0.42]; [0.41, 0.76], [0.71, 0.40])$
$\tilde{z}_3^{(3)} = ([0.51, 0.57, 0.25, 0.26]; [0.61, 0.72], [0.69, 0.56])$
$\tilde{z}_3^{(4)} = ([0.48, 0.45, 0.33, 0.45]; [0.44, 0.75], [0.75, 0.42])$

$\tilde{z}_4^{(1)} = ([0.53, 0.34, 0.34, 0.33]; [0.49, 0.73], [0.75, 0.43])$
$\tilde{z}_4^{(2)} = ([0.40, 0.43, 0.45, 0.52]; [0.52, 0.71], [0.79, 0.57])$
$\tilde{z}_4^{(3)} = ([0.41, 0.57, 0.42, 0.50]; [0.52, 0.78], [0.69, 0.41])$
$\tilde{z}_4^{(4)} = ([0.43, 0.58, 0.47, 0.37]; [0.55, 0.62], [0.63, 0.62])$

Step 4: Applying the known weight vector by using operational law 3 in definition 7, and score function to order the overall preference interval Pythagorean trapezoidal fuzzy values such that,

$\tilde{z}_1^{(1)} = ([0.52, 0.35, 0.64, 0.54]; [0.67, 0.50], [0.66, 0.35])$
$\tilde{z}_1^{(2)} = ([0.44, 0.37, 0.55, 0.34]; [0.46, 0.14], [0.75, 0.44])$
$\tilde{z}_1^{(3)} = ([0.31, 0.35, 0.36, 0.28]; [0.53, 0.51], [0.51, 0.71])$
$\tilde{z}_1^{(4)} = ([0.57, 0.26, 0.24, 0.15]; [0.28, 0.74], [0.63, 0.64])$

$\tilde{z}_2^{(1)} = ([0.37, 0.29, 0.45, 0.44]; [0.36, 0.74], [0.72, 0.53])$
$\tilde{z}_2^{(2)} = ([0.51, 0.68, 0.56, 0.75]; [0.80, 0.40], [0.75, 0.48])$
$\tilde{z}_2^{(3)} = ([0.56, 0.44, 0.68, 0.66]; [0.43, 0.64], [0.82, 0.39])$
$\tilde{z}_2^{(4)} = ([0.60, 0.61, 0.60, 0.75]; [0.56, 0.47], [0.67, 0.45])$

$\tilde{z}_3^{(1)} = ([0.28, 0.27, 0.19, 0.27]; [0.33, 0.84], [0.62, 0.19])$
$\tilde{z}_3^{(2)} = ([0.71, 0.79, 0.35, 0.36]; [0.69, 0.59], [0.77, 0.44])$
$\tilde{z}_3^{(3)} = ([0.74, 0.63, 0.49, 0.50]; [0.44, 0.66], [0.75, 0.33])$
$\tilde{z}_3^{(4)} = ([0.32, 0.32, 0.40, 0.39]; [0.35, 0.69], [0.68, 0.54])$

$\tilde{z}_4^{(1)} = ([0.42, 0.27, 0.27, 0.26]; [0.44, 0.79], [0.67, 0.50])$
$\tilde{z}_4^{(2)} = ([0.48, 0.56, 0.54, 0.62]; [0.56, 0.75], [0.75, 0.50])$
$\tilde{z}_4^{(3)} = ([0.25, 0.34, 0.28, 0.22]; [0.44, 0.75], [0.50, 0.75])$
$\tilde{z}_4^{(4)} = ([0.57, 0.79, 0.58, 0.07]; [0.59, 0.59], [0.85, 0.28])$

Step 5: Applying the $(I-IPTFHA)$ operator to originate the mutual inclusive predilection interval Pythagorean trapezoidal fuzzy values $\tilde{z}_i$. Consider that,

$(w=0.20, 0.30, 0.35, 0.15)$ and,

$\Gamma = (0.155, 0.345, 0.345, 0.155)$

$\tilde{z}_1 = ([0.36, 0.28, 0.36, 0.38]; [.44, .75], [.66, .34])$
$\tilde{z}_2 = ([0.56, 0.65, 0.48, 0.53]; [.70, .54], [.75, .46])$
$\tilde{z}_3 = ([0.53, 0.47, 0.57, 0.47]; [.46, .65], [.68, .39])$
$\tilde{z}_4 = ([0.49, 0.48, 0.47, 0.52]; [.49, .63], [.69, .47])$

Step 6: Applying score function $s(\tilde{z}_i)$ of the interval Pythagorean trapezoidal fuzzy numbers such that, $s(z_1)=0.035, s(z_2)=-0.002, s(z_3)=-0.004, s(z_4)=-0.014$

Step 7: We determine the Rank of all alternatives according to the Score function $s(\tilde{z}_i)$ of the interval Pythagorean trapezoidal fuzzy numbers $s(\tilde{z}_i)$ }, hence $B_1 \geq B_4 \geq B_2 \geq B_3$. B_1 is best alternative (Fig. 2).

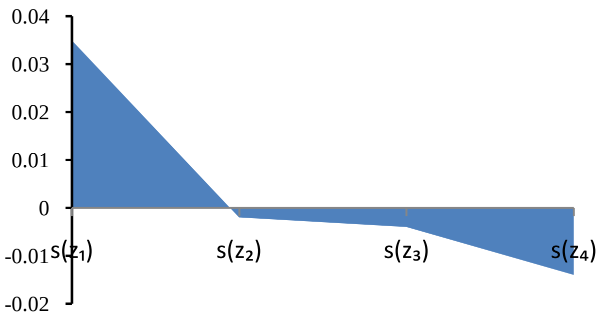

Fig. 2: Graph of the best alternative

6. Conclusions

In this paper we introduced the idea of IPTFWA, I-IPTFOWA and I-IPTFHA operators. we have defined some appropriate properties such as, monotonicity, idem potency and bounded of I-IPTFOWA and I-IPTFHA operators. We also developed aggregation I-IPTFHA operator, which is a generalization of the I-IPTFOWA operators. At the end of this paper we have constructed numerical an application of I-IPTFOWA and I-IPTFHA operators to MAGDM problems with interval Pythagorean trapezoidal fuzzy information. In future we can extend this work to other different fields.

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