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On Bipolar-Valued Hesitant Fuzzy Sets and Their Applications in Multi-Attribute Decision Making

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ABSTRACT

In this manuscript, a novel structure of bipolar-valued hesitant fuzzy set (BPVHFS) is proposed as a generalization of fuzzy set (FS). The basic set theoretic operations of BPVHFSs are defined and their related results are studied. Based on the basic operations, some aggregation operators for BPVHFSs are developed and their fitness is verified using principle of mathematical induction. The multi-attribute decision making (MADM) is established in the framework of BPVHFSs and a numerical example is provided to illustrate the process. The article ended with some concluding remarks along with some future directions.

1. Introduction

The work-room of decision making has a stretched past. It is among one of the greatly respected research fields now days. Almost everyone in their living must take decision for example to select best place or to select best education system or to select a certain business. Decision making problems are not able to be got answers by normal mathematical techniques.

Zadeh [1] in 1965 developed the idea of FS in order to deal with different kinds of uncertainties. After this idea of FSs several types of extensions of FSs came into existence including bipolar-valued fuzzy set (BVFS) [2], inter-valued fuzzy set (IVFS) [3] and intuitionistic fuzzy set (IFS) [4] etc. are some well-known sets. In decision making process when we need the aggregation of fuzzy information, we need to look at the basic properties of all these advanced forms of FSs. Work on the properties of FSs and their advanced form has already been done.

In the area of FSs, alternatives that have a grade 0 has no satisfaction at all while those alternatives that have a grade 1 has full satisfaction and those having degree between 0 and 1 are considered as to have partial satisfaction. For this reason, common FSs did not discuss the case of disagreement. So, keeping in view this idea, Lee [2] gives greater value to the idea of FSs by enlarging the interval from $[0, 1]$ to $[-1, 1]$ to introduce a new enhanced form of FS known as BVFS. After this, many scientists took keen interest in BVFSs. In 2008, Rui and Pierre [5] described MADM process for BVFSs. Mahmood and Munir [6] developed bipolar fuzzy subgroups. Similarly, Aslam et al. [7] gives the concept of bipolar fuzzy soft sets and applied it in MADM. BVFSs have special positive and negative grades for the satisfactory and dissatisfactory level of an available alternative. This sort of fuzzy algebraic structure offers two-sided DM for example good quality and bad quality, like and not like and so on. In almost every field of living, BVFSs has a great importance like social, medical, business and management sciences and so on.

In the work-room of FSs the hardest step is to assign affiliations degrees (membership degrees) of elements to sets. Several steps were taken to make less these difficulties and very shortly Torra and Narukawa [8] and Torra [9] presented the theory of hesitant fuzzy set (HFS). The point or amount unlike is that HFSs have affiliation degrees in the form of few elements. This was a new move and supporting this new move further they defined several operations for HFSs including the measurement of HFSs by defining score function. Chen et al. [10] developed a function for the deviation degree HFSs and developed the idea of correlation coefficients between HFSs and made them useful for controlling clustering. HFSs were further used by Xu [11] in MADM process. For some related work in MADM one is refer to [12-23].

When fuzzy information is in the form of finite subsets of $[0, 1]$, we use HFSs. Sometime affiliation functions have the membership grades in the interval $[0, 1]$ and $[-1, 0]$. In such a case, we used BVFSs. Consider a situation where we are given fuzzy information in the form of finite subsets of $[0, 1]$& $[-1, 0]$. To deal with such a situation we need to develop a new structureto fulfill our requirements.

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In this manuscript, we merged HFS and BVFS and developed a generalization of FSs known as BPVHFSs. A BVFS has affiliation function whose membership grades lies in the interval $[0, 1]$ and $[-1, 0]$. While a BPVHFS has affiliation function in terms of finite subsets of $[0, 1]$ and $[-1, 0]$. The membership and non-membership functions are hesitant fuzzy elements (HFEs) in the form of finite subsets of $[0, 1]$ and $[-1, 0]$. We also defined bipolar-valued HFE (BPVHFE). The various types of basic operations for BPVH FSs are defined and the properties of these operations are studied. To aggregate the BPVHFS, we defined some aggregation operators for BPVHFS and then applied these aggregation operators MADM process. To explain the defined operations, a numerical example is solved.

This manuscript consists of five sections. Section one provided a brief introduction and background of proposed theory. In section two, we discussed some pre-requisites of the new proposed structure. The ideas of HFSs and BVFSs are discussed with their basic operations. In section three, the new structure of BPVHFSs and BPVHFEs are proposed and their basic operations are defined. The fourth section consists of aggregation operators for BPVHFSs along with their properties and an example. In section five, we defined MADM technique and then using defined aggregation operators we solved a MADM problem. Finally, some advantages of proposed work are discussed and concluding remarks are added.

2. Preliminaries

This section consists of some pre-requisites of BPVHFSs. As BPVHFS is a combination of BVFS and HFSs so in this section first we recall the definition and some useful properties of BVFSs. We also recall the concepts of HFSs and their properties.

Definition 1: [2]

For any set X, a BVFSA of X is expressed in the form of : $\mathrm{B} = \{(\mathrm{m}, (F^p(\mathrm{m}), F^n(\mathrm{m})) : \mathrm{m} \in \mathrm{X}\}$.

Where $F^p : \mathrm{X} \to [0,1]$ and $F^n : \mathrm{X} \to [-1,0]$ are mappings such as $F^p : X \to [0,1]$ denotes the intensity of how much an element m obey a property corresponding to the BVFSBand $F^n : X \to [-1,0]$ denotes the intensity of how an element m offer opposition to some property corresponding to the BVFSB. If $F^p(m) \neq 0$ and $F^n(m) = 0$, in this case m is regarded as to obey a property corresponding to the BVFSB. If $F^p(m) = 0$ and $F^n(m) \neq 0$,in this case $m$is regarded as not to obey the property corresponding to the BVFSB. When $F^p(m) \neq 0$ and $F^n(m) \neq 0$ then there is an overlap between the affiliation functions of the properties.

ABVFS B on a set X is also of the form: $\mathrm{B} = \{(m, F^R(m)) : m \in \mathrm{X}\}$ where $F^R : \mathrm{X} \to [-1,1]$ defined by

$$F^R(m) = \begin{cases} F^p(m) \ if \ \ F^n(m) = 0 \\ F^n(m) \ if \ \ F^p(m) = 0 \\ \mu(F^p(m), F^n(m)) \ else \end{cases}$$

Here $\mu(F^p(m),\ F^-(m))$ is an aggregation function and it may be defined in enormous ways. Its choice depends on the application domain.

For any two BVFSs

$$\widetilde{\mathrm{A}} = \{(\mathrm{m}, (\mathrm{F^p}_{\widetilde{\mathrm{A}}}(\mathrm{m}), (\mathrm{F^n}_{\widetilde{\mathrm{A}}}(\mathrm{m}))) : \mathrm{m} \in \mathrm{X}\}$$

$$\mathrm{B} = \{(\mathrm{m}, (\mathrm{F^p}_{\mathrm{B}}(\mathrm{m}), (\mathrm{F^n}_{\mathrm{B}}(\mathrm{m}))) : \mathrm{m} \in \mathrm{X}\}$$

The union and intersection of two BVFSs and the complement of a BVFS are defined as:

$\widetilde{\mathrm{A}} \cup \mathrm{B} = \{(\mathrm{m}, \mathrm{F}_{\widetilde{\mathrm{A}}\cup\mathrm{B}}(\mathrm{m}) | \mathrm{m} \in \mathrm{X}\}$.

$F_{\widetilde{\mathrm{A}}\cup\mathrm{B}}(m) = \{F^p{}_{\widetilde{\mathrm{A}}\cup\mathrm{B}}(m), F^n{}_{\widetilde{\mathrm{A}}\cup\mathrm{B}}(m)\}$.

$F^p{}_{\widetilde{\mathrm{A}}\cup\mathrm{B}}(m) = \max\{F^p{}_{\widetilde{\mathrm{A}}}(m), F^p{}_B(m)\}$.

$F^n{}_{\widetilde{\mathrm{A}}\cup\mathrm{B}}(m) = \min\{F^n{}_{\widetilde{\mathrm{A}}}(m), F^n{}_B(m)\}$.

$\widetilde{\mathrm{A}} \cap \mathrm{B} = \{(m, F_{\widetilde{\mathrm{A}}\cap\mathrm{B}}(m) | m \in \mathrm{X}\}$.

$F_{\widetilde{\mathrm{A}}\cap\mathrm{B}}(m) = \{F^p{}_{\widetilde{\mathrm{A}}\cap\mathrm{B}}(m), F^n{}_{\widetilde{\mathrm{A}}\cap\mathrm{B}}(m)\}$.

$F^p{}_{\widetilde{\mathrm{A}}\cap\mathrm{B}}(m) = \min\{F^p{}_{\widetilde{\mathrm{A}}}(m), F^p{}_B(m)\}$.

$F^n{}_{\widetilde{\mathrm{A}}\cap\mathrm{B}}(m) = \max\{F^n{}_{\widetilde{\mathrm{A}}}(m), F^n{}_B(m)\}$.

$\overline{\widetilde{\mathrm{A}}} = \{(m, F_{\overline{\widetilde{\mathrm{A}}}}(m)) | m \in \mathrm{X}\}$.

$F_{\overline{\widetilde{\mathrm{A}}}}(m) = \{F^p{}_{\overline{\widetilde{\mathrm{A}}}}(m),\ F^n{}_{\overline{\widetilde{\mathrm{A}}}}(m)\}$.

$F^p{}_{\overline{\widetilde{\mathrm{A}}}}(m) = 1 - F^p{}_{\widetilde{\mathrm{A}}}(m)$.

$F^n{}_{\overline{\widetilde{\mathrm{A}}}}(m) = -1 - F^n{}_{\widetilde{\mathrm{A}}}(m)$.

Definition 2: [8, 9]

Let X is any set. Then a HFS on Xin the form of a function F that when applied to Xgives usfew values in $[0, 1]$. A HFSH on X is denoted as $H = \{\langle m, \mathrm{F}(m)\rangle | \forall m \in \mathrm{X}\}$, where $\mathrm{F}(m)$ is a set of few distinct values in the interval $[0,1]$ representing the possible affiliation degree of the element $m \in \mathrm{X}$ to H. The set of all HFSs is expressed by$H(m) = \{\{\langle m, \mathrm{F}(m)\rangle | \ \forall m \in \mathrm{X}\}\}$. Here $\mathrm{F}(m)$ is known as HFE, the primary unit of HFS.

Some operation on, HFSs are given as follows:

LB:$\mathrm{F^n}(\mathrm{m}) = \min \mathrm{F}(\mathrm{m})$

α−LB:$\mathrm{F}^{\mathrm{n}}_{\alpha}(\mathrm{m}) = \{\mathrm{S} \in \mathrm{F}(\mathrm{m}) : \mathrm{S} \leq \alpha\}$

UB:$\mathrm{F^p}(\mathrm{m}) = \max \mathrm{F}(\mathrm{m})$

α−UB: $\mathrm{F}^{\mathrm{p}}_{\alpha}(\mathrm{m}) = \{\mathrm{s} \in \mathrm{F}(\mathrm{m}) : \mathrm{s} \geq \alpha\}$

Complement: $\mathrm{F^c}(\mathrm{m}) = \{1 - \mathrm{s} : \mathrm{s} \in \mathrm{F}(\mathrm{m})\}$

Union:

$$(F_1 \cup F_2)(m) = \left\{\begin{matrix} s \in F_1(m) \cup F_2(m): \\ s \geq \max\{F_1^n(m), F_2^n(m)\} \end{matrix}\right\}$$

Intersection:

$$(F_1 \cap F_2)(m) = \left\{\begin{matrix} s \in F_1(m) \cup F_2(m): \\ s \leq \min\{F_1^p(m), F_2^p(m)\} \end{matrix}\right\}$$

$$F^{\rho^{\cdot}}(m) = \{s^{\rho^{\cdot}}: s \in F(m)\}$$

$$\rho^{\cdot}F(m) = \{1-(1-s)^{\rho^{\cdot}}: s \in F(m)\}$$

$$(F_1 \oplus F_2)(m) = \left\{\begin{matrix} s_1 + s_2 - s_1 s_2: \\ s_1 \in F_1(m), s_2 \in F_2(m) \end{matrix}\right\}$$

$$(F_1 \otimes F_2)(m) = \{s_1 s_2: s_1 \in F_1(m), s_2 \in F_2(m)\}$$

3. Results

When fuzzy information is in the form of finite subsets of $[0,1]$, we use hesitant fuzzy sets. If affiliation functions have the membership grades in the interval $[0,1]$ and $[-1,0]$, bipolar-valued fuzzy sets are used to deal with it. Consider a situation where we are given fuzzy information in the form of finite subsets of $[0,1]$& $[-1,0]$. To deal with such a situation we need to develop the concept of bipolar-valued hesitant fuzzy sets.

Definition 3:

For any set X, the BPVHFSB on some domain of X is denoted and defined by:

$$B = \{(m, (F^+(m), F^-(m))): m \in X\}$$

Where $F^+: X \to [0,1]$ and $F^-: X \to [-1,0]$ are HFEs denoting the membership and non-membership grade of element "m" corresponding to BPVHFSB. Here $H = \{F^+(m), F^-(m)\}$ is a BPVHFE.

Consider two BPVHFSs.

$$\widetilde{A} = \{(m, (F^+{}_{\widetilde{A}}(m), (F^-{}_{\widetilde{A}}(m))): m \in X\}$$

$$B = \{(m, (F^+{}_B(m), (F^-{}_B(m))): m \in X\}$$

Then we have:

$$\widetilde{A} \cup B = \left\{\begin{matrix} s \in F_{\widetilde{A}}(m) \cup F_B(m): \\ (F^+{}_{\widetilde{A}} \cup F^+{}_B)(m), (F^-{}_{\widetilde{A}} \cup F^-{}_B)(m) \end{matrix}\right\}.$$

$$(F^+{}_{\widetilde{A}} \cup F^+{}_B)(m) = \left\{\begin{matrix} s \in F^+{}_A(m) \cup F^+{}_B(m): \\ s \geq max\{F^+{}_{\widetilde{A}}{}^-(m), F^+{}_B{}^-(m)\} \end{matrix}\right\}.$$

$$(F^-{}_{\widetilde{A}} \cup F^-{}_B)(m) = \left\{\begin{matrix} s \in F^-{}_{\widetilde{A}}(m) \cup F^-{}_B(m): \\ s \leq min\{F^-{}_{\widetilde{A}}{}^+(m), F^-{}_B{}^+(m)\} \end{matrix}\right\}.$$

$$\widetilde{A} \cap B = \left\{\begin{matrix} s \in F_{\widetilde{A}}(m) \cup F_B(m): \\ (F^+{}_{\widetilde{A}} \cap F^+{}_B)(m), (F^-{}_{\widetilde{A}} \cap F^-{}_B)(m) \end{matrix}\right\}.$$

$$(F^+{}_{\widetilde{A}} \cap F^+{}_B)(m) = \left\{\begin{matrix} s \in F^+{}_{\widetilde{A}}(m) \cup F^+{}_B(m): \\ s \leq min\{F^+{}_{\widetilde{A}}{}^+(m), F^+{}_B{}^+(m)\} \end{matrix}\right\}.$$

$$(F^-{}_{\widetilde{A}} \cap F^-{}_B)(m) = \left\{\begin{matrix} s \in F^-{}_{\widetilde{A}}(m) \cup F^-{}_B(m): \\ s \geq max\{F^-{}_{\widetilde{A}}{}^-(m), F^-{}_B{}^-(m)\} \end{matrix}\right\}.$$

$$(\widetilde{A})^c = \{(m, (F^+{}_{\widetilde{A}}(m))^c, (F^-{}_{\widetilde{A}}(m))^c): m \in X\}.$$

$$(F^+{}_{\widetilde{A}}(m))^c = \{1 - s: s \in F^+{}_{\widetilde{A}}(m)\}.$$

$$(F^-{}_{\widetilde{A}}(m))^c = \{-1 - s: s \in F^-{}_{\widetilde{A}}(m)\}.$$

$$(\widetilde{A} \oplus B)(m) = \{s_1 + s_2 - s_1 s_2: s_1 \in F^+{}_{\widetilde{A}}(m), s_2 \in F^+{}_B(m), -(s_1 s_2): s_1 \in F^-{}_{\widetilde{A}}(m), s_2 \in F^-{}_B(m)\}.$$

$$(\widetilde{A} \otimes B)(m) = \{s_1 s_2 : s_1 \in F^+{}_{\widetilde{A}}(m), s_2 \in F^+{}_B(m), -(-s_1 - s_2 - s_1 s_2): s_1 \in F^-{}_{\widetilde{A}}(m), s_2 \in F^-{}_B(m)\}.$$

For any $\rho^{\cdot} > 0$

$$\rho^{\cdot}\widetilde{A}(m) = \{1 - (1-s)^{\rho^{\cdot}}: s \in F^+{}_{\widetilde{A}}(m), -(-s^{\rho^{\cdot}}): s \in F^-{}_{\widetilde{A}}(m)\}.$$

$$\widetilde{A}^{\rho^{\cdot}}(m) = \{s^{\rho^{\cdot}}: s \in F^+{}_{\widetilde{A}}(m), -1 - \left(-\left(-(-1-s)\right)^{\rho^{\cdot}}\right): s \in F^-{}_{\widetilde{A}}(m)\}.$$

Some special forms of BPVHFSs are:

The set $B^o = \{(m, (F^{o+}(m), F^{o-}(m))): m \in X\}$ represents empty BPVHFS where

$$F^{o+}(m) = \{0\}\ \forall\, m \in X \ and\ F^{o-}(m) = \{-1\}\ \forall\, m \in X.$$

The set $B^f = \{(m, (F^{f^+}(m), F^{f^-}(m))): m \in X\}$ represents full BPVHFS where

$$F^{f^+}(m) = \{1\}\ \forall\, m \in X \text{ and } F^{f^-}(m) = \{0\}\ \forall\, m \in X.$$

The complete ignorance set is represented by $B^u = \{(m, (F^{u+}(m), F^{u-}(m))): m \in X\}$ Where

$$F^{u+}(m) = [0,1]\ \forall\, m \in X \text{ and} F^{u-}(m) = [-1,0]\ \forall\, m \in X.$$

The set $B^N = \{(m, (F^{N^+}(m), F^{N^-}(m))): m \in X\}$is a nonsense set where

$$F^{N^+}(m) = \{\quad\}\ \forall\, m \in X\ and\ F^{N^-}(m) = \{\quad\}\ \forall\, m \in X.$$

Example 1:

Let $X = \{m_1, m_2\}$ and

$$\widetilde{A} = \left\{\begin{matrix} < m_1, \{0.1, 0.2\}, \{-0.3, -0.2\} >, \\ < m_2, \{0.4, 0.5\}, \{-0.6, -0.5\} > \end{matrix}\right\}$$

$$B = \left\{\begin{matrix} < m_1, \{0.3, 0.4\}, \{-0.5, -0.4\} >, \\ < m_2, \{0.5, 0.6\}, \{-0.7, -0.6\} > \end{matrix}\right\}.$$

$$\widetilde{A} \cup B = \left\{\begin{matrix} < m_1, \{0.3, 0.4\}, \{-0.5, -0.4\} >, \\ < m_2, \{0.5, 0.6\}, \{-0.7, -0.6\} > \end{matrix}\right\}.$$

$$\widetilde{A} \cap B = \left\{\begin{matrix} < m_1, \{0.1, 0.2\}, \{-0.2, -0.3\} >, \\ < m_2, \{0.4, 0.5\}, \{-0.5, -0.6\} > \end{matrix}\right\}.$$

$$\widetilde{A}^c = \left\{\begin{matrix} < m_1, \{0.9, 0.8\}, \{-0.7, -0.8\} >, \\ < m_2, \{0.5, 0.6\}, \{-0.5, -0.4\} > \end{matrix}\right\}.$$

$\left(\widetilde{A}\oplus B\right)(m) =$

$\begin{cases} < m_1, \{0.37, 0.46, 0.44, 0.52\}, \{-0.12, -0.15, -0.08, -0.1\} >, \\ < m_2, \{0.7, 0.76, 0.75, 0.8\}, \{-0.36, -0.42, -0.3, -0.35\} > \end{cases}$

$\left(\widetilde{A}\otimes B\right)(m) =$

$\begin{cases} < m_1, \{0.03, 0.04, 0.06, 0.08\}, \{-0.58, -0.65, -0.52, -0.6\} >, \\ < m_2, \{0.2, 0.24, 0.25, 0.3\}, \{-0.88, -0.84, -0.85, -0.8\} > \end{cases}$

For $\rho^{\cdot} = 2$

$\rho^{\cdot}\widetilde{A}(m) = \begin{cases} < m_1, \{0.19, 0.36\}, \{-0.09, -0.04\} >, \\ < m_2, \{0.64, 0.75\}, \{-0.36, -0.25\} > \end{cases}.$

$\widetilde{A}^{\rho^{\cdot}}(m) = \begin{cases} < m_1, \{0.01, 0.04\}, \{-0.51, -0.36\} >, \\ < m_2, \{0.16, 0.25\}, \{-0.84, -0.75\} > \end{cases}.$

Theorem 1:

For BPVHFEs H, H_A, H_B and $\rho^{\cdot}, \rho^{\cdot}_1, \rho^{\cdot}_2 > 0$.

1. $H^c_{\widetilde{A}} \cup H^c_B = (H_{\widetilde{A}} \cap H_B)^c$
2. $H^c_{\widetilde{A}} \cap H^c_B = (H_{\widetilde{A}} \cup H_B)^c$
3. $(H^c)^{\rho^{\cdot}} = (\rho^{\cdot}H)^c$
4. $\rho^{\cdot}(H^c) = (H^{\rho^{\cdot}})^c$
5. $H^c_{\widetilde{A}} \oplus H^c_B = (H_{\widetilde{A}} \otimes H_B)^c$
6. $H^c_{\widetilde{A}} \otimes H^c_B = (H_{\widetilde{A}} \oplus H_B)^c$
7. $H_{\widetilde{A}} \oplus H_B = H_B \oplus H_{\widetilde{A}}$
8. $H_{\widetilde{A}} \otimes H_B = H_B \otimes H_{\widetilde{A}}$
9. $\rho^{\cdot}(H_{\widetilde{A}} \oplus H_B) = \rho^{\cdot}H_{\widetilde{A}} \oplus \rho^{\cdot}H_B$
10. $H^{\rho^{\cdot}}_{\widetilde{A}} \otimes H^{\rho^{\cdot}}_B = (H_{\widetilde{A}} \otimes H_B)^{\rho^{\cdot}}$
11. $(\rho^{\cdot}_1\rho^{\cdot}_2)H = \rho^{\cdot}_1(\rho^{\cdot}_2 H)$
12. $H^{\rho^{\cdot}_1\rho^{\cdot}_2} = (H^{\rho^{\cdot}_1})^{\rho^{\cdot}_2}$

Proof

We prove the result 1, 3, 5, 7, 9, 11 and 12. The remaining results can be proved similarly.

1. $H^c_{\widetilde{A}} \cup H^c_B = \begin{Bmatrix} \{1 - s_1\}, \\ \{-1 - s_1\} \end{Bmatrix} \cup \begin{Bmatrix} \{1 - s_2\}, \\ \{-1 - s_2\} \end{Bmatrix}.$

$= \bigcup_{\gamma_1 \in H_{\widetilde{A}}, s_2 \in H_B} \begin{Bmatrix} \{max\{1 - s_1, 1 - s_2\}\}, \\ \{min\{-1 - s_1, -1 - s_2\}\} \end{Bmatrix}.$

$= \bigcup_{\gamma_1 \in H_{\widetilde{A}}, s_2 \in H_B} \begin{Bmatrix} \{1 - min\{s_1, s_2\}\}, \\ \{-1 - max\{-s_1 s_2\}\} \end{Bmatrix}.$

$= (H_{\widetilde{A}} \cap H_B)^c$

2. $(H^c)^{\rho^{\cdot}} = \begin{Bmatrix} \{(1 - s)^{\dot{\rho}}\}, \\ \left\{-1 - \left(-\left(-(-1 - (-1 - s))\right)^{\rho^{\cdot}}\right)\right\} \end{Bmatrix}$

$= \left\{\{(1 - s)^{\rho^{\cdot}}\}, \left\{-1 - \left(-(-s^{\rho^{\cdot}})\right)\right\}\right\}$

$\rho^{\cdot}H = \left\{\{1 - (1 - s)^{\rho^{\cdot}}\}, \{-(-s^{\rho^{\cdot}})\}\right\}$

$(\rho^{\cdot}H)^c = \left\{\{1 - (1 - (1 - s))^{\rho^{\cdot}}\}, \left\{-1 - \left(-(-s^{\rho^{\cdot}})\right)\right\}\right\}$

$= \left\{\{(1 - s)^{\rho^{\cdot}}\}, \left\{-1 - \left(-(-s^{\rho^{\cdot}})\right)\right\}\right\}$

Hence $(H^c)^{\rho^{\cdot}} = (\rho^{\cdot}H)^c$

5. $H^c_{\widetilde{A}} \oplus H^c_B = \begin{matrix} \{\{1 - s_1\}, \{-1 - s_1\}\} \oplus \\ \{\{1 - s_2\}, \{-1 - s_2\}\} \end{matrix}$

$= \begin{Bmatrix} \{(1 - s_1) + (1 - s_2) - (1 - s_1)(1 - s_2)\}, \\ \{-(-1 - s_1)(-1 - s_2)\} \end{Bmatrix}.$

$= \{\{1 - s_1 s_2\}, \{-(1 + s_1 + s_2 + s_1 s_2)\}\}.$

$H_{\widetilde{A}} \otimes H_B = \{\{s_1 s_2\}, \{-(-s_1 - s_2 - s_1 s_2)\}\}.$

$(H_{\widetilde{A}} \otimes H_B)^c = \{\{1 - s_1 s_2\}, \{-1 - [-(-s_1 - s_2 - s_1 s_2)]\}\}.$

$= \{\{1 - s_1 s_2\}, \{-(1 + s_1 + s_2 + s_1 s_2)\}\}$

Hence $H^c_{\widetilde{A}} \oplus H^c_B = (H_{\widetilde{A}} \otimes H_B)^c$

$H_{\widetilde{A}} \oplus H_B = \{\{s_1 + s_2 - s_1 s_2\}, \{-(s_1 s_2)\}\}$

$= \{\{s_2 + s_1 - s_2 s_1\}, \{-(s_2 s_1)\}\}$

$= H_B \oplus H_{\widetilde{A}}.$

9. $\rho^{\cdot}(H_{\widetilde{A}} \oplus H_B) = \rho^{\cdot}\{\{s_1 + s_2 - s_1 s_2\}, \{-(s_1 s_2)\}\}$

$= \begin{Bmatrix} \left\{1 - \left(1 - (s_1 + s_2 - s_1 s_2)\right)^{\dot{\rho}}\right\}, \\ \left\{-\left[-\left(-(s_1 s_2)\right)^{\rho^{\cdot}}\right]\right\} \end{Bmatrix}.$

$= \left\{\{1 - (1 - s_1)^{\rho^{\cdot}}(1 - s_2)^{\rho^{\cdot}}\}, \{-(s_1 s_2)^{\rho^{\cdot}}\}\right\}.$

$\rho^{\cdot}H_{\widetilde{A}} \oplus \rho^{\cdot}H_B = \begin{matrix} \left\{\{1 - (1 - s_1)^{\rho^{\cdot}}\}, \{-(-s_1)^{\rho^{\cdot}}\}\right\} \oplus \\ \left\{\{1 - (1 - s_2)^{\rho^{\cdot}}\}, \{-(-s_2)^{\rho^{\cdot}}\}\right\} \end{matrix}$

$= \begin{Bmatrix} \begin{Bmatrix} \left(1 - (1 - s_1)^{\dot{\rho}}\right) + \left(1 - (1 - s_2)^{\dot{\rho}}\right) - \\ \left(\left(1 - (1 - s_1)^{\dot{\rho}}\right)\right)\left(1 - (1 - s_2)^{\dot{\rho}}\right) \end{Bmatrix}, \\ \{-(-(-s_1)^{\rho^{\cdot}})(-(-s_2)^{\rho^{\cdot}})\} \end{Bmatrix}$

$= \left\{\{1 - (1 - s_1)^{\rho^{\cdot}}(1 - s_2)^{\rho^{\cdot}}\}, \{-(s_1 s_2)^{\rho^{\cdot}}\}\right\}.$

Hence $\rho^{\cdot}(H_{\widetilde{A}} \oplus H_B) = \rho^{\cdot}H_{\widetilde{A}} \oplus \rho^{\cdot}H_B$

11. $(\rho^{\cdot}_1\rho^{\cdot}_2)H = \left\{\{1 - (1 - s)^{\rho^{\cdot}_1\rho^{\cdot}_2}\}, \{-(-s)^{\rho^{\cdot}_1\rho^{\cdot}_2}\}\right\}$

$\rho^{\cdot}_1(\rho^{\cdot}_2 H) = \rho^{\cdot}_1\left\{\{1 - (1 - s)^{\rho^{\cdot}_2}\}, \{-(-s)^{\rho^{\cdot}_2}\}\right\}$

$= \begin{Bmatrix} \{1 - ((1 - 1 + (1 - s)^{\dot{\rho}_2})^{\dot{\rho}_1}\}, \\ \left\{-\left(-(-(-s)^{\rho^{\cdot}_2})^{\rho^{\cdot}_1})\right)\right\} \end{Bmatrix}$

$= \left\{\{1 - (1 - s)^{\rho^{\cdot}_1\rho^{\cdot}_2}\}, \{-(-s)^{\rho^{\cdot}_1\rho^{\cdot}_2}\}\right\}.$

Hence $(\rho^{\cdot}{}_1\rho^{\cdot}{}_2)\mathrm{H} = \rho^{\cdot}{}_1(\rho^{\cdot}{}_2\mathrm{H})$

12. $\mathrm{H}^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2} = \left\{\{s^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2}\}, \left\{-1-\left(-\left(-(-1-s)\right)^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2}\right)\right\}\right\}$

$= \left\{\{s^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2}\}, \{-1-(-(1+s)^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2})\}\right\}.$

$(\mathrm{H}^{\rho^{\cdot}{}_1})^{\rho^{\cdot}{}_2} = \left\{\{s^{\rho^{\cdot}{}_1}\}, \left\{-1-\left(-\left(-(-1-s)\right)^{\rho^{\cdot}{}_1}\right)\right\}\right\}^{\rho^{\cdot}{}_2}$

$$= \left\{\begin{matrix} \{(s^{\dot{\rho}_1})^{\dot{\rho}_2}\}, \\ \left\{-1-\left(-\left(-\left(-1-\left(-\left(-(-1-s)\right)^{\rho^{\cdot}{}_1}\right)\right)^{\rho^{\cdot}{}_2}\right)\right)\right\} \end{matrix}\right\}$$

$= \left\{\{s^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2}\}, \left\{-1+\left(-\left(-1+1-(1+s)\right)^{\rho^{\cdot}{}_1}\right)^{\rho^{\cdot}{}_2}\right\}\right\}.$

$= \left\{\{s^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2}\}, \{-1-(-(1+s)^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2})\}\right\}.$

Hence $(\mathrm{H}^{\rho^{\cdot}{}_1\rho^{\cdot}{}_2} = (\mathrm{H}^{\rho^{\cdot}{}_1})^{\rho^{\cdot}{}_2}$

2. Aggregation Operators

In this section, the aggregation operators for BPVHFSs are developed including BPVHFWA operator and BPVHFWG operator. These aggregation operators are described with the help of examples. These aggregation operators are then successfully applied in MADM problems in next section.

Definition 4:

Let $H_i(i = 1,2,3,4 \ldots n)$ be a set of BPVHFEs and let $\omega' = (\omega'_1, \omega'_2, \ldots \omega'_n)^T$ be the weight vector of $\mathrm{H}_i(i = 1,2,3,4 \ldots n)$ with $\omega'_i \in [0,1]$ and $\sum_{i=1}^{n} \omega'_i = 1$. Then the BPVHFWA operator is a function $\Psi^n \to \Psi$ such that

$$BPVHFWA(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n) = \bigoplus_{i=1}^{n} (\omega'_i \mathrm{H}_i)$$

Theorem 2:

The aggregated value of BPVHFEs $H_i(i = 1,2,3,4 \ldots n)$ using BPVHFWA operator is a BPVHFE and \$BPVHFWA(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n)$

$$= \left\{\begin{matrix} \left\{1-\prod_{i=1}^{n}(1-s_i)^{\acute{\omega}_i}: s_i \in \mathrm{H}_i^+\right\}, \\ \left\{-\prod_{i=1}^{n}(-s_i)^{\omega'_i}: s_i \in \mathrm{H}_i^-\right\} \end{matrix}\right\} \ldots (*)$$

Proof:

The first part of this theorem is obvious. To prove the second part, we need to use mathematical induction.

Case 1: When $n = 2$

$$\omega'_1\mathrm{H}_1 = \left\{\begin{matrix} \{1-(1-s_1)^{\acute{\omega}_1}: s_1 \in \mathrm{H}_1^+\}, \\ \{-(-s_1)^{\omega'_1}: s_1 \in \mathrm{H}_1^-\} \end{matrix}\right\}.$$

$$\omega'_2\mathrm{H}_2 = \left\{\begin{matrix} \{1-(1-s_2)^{\acute{\omega}_2}: s_2 \in \mathrm{H}_2^+\}, \\ \{-(-s_2)^{\omega'_2}: s_2 \in \mathrm{H}_2^-\} \end{matrix}\right\}.$$

Now $\omega'_1\mathrm{H}_1 \oplus \omega'_2\mathrm{H}_2$

$$= \begin{matrix} \left\{\{1-(1-s_1)^{\acute{\omega}_1}: s_1 \in \mathrm{H}_1^+\}, \{-(-s_1)^{\omega'_1}: s_1 \in \mathrm{H}_1^-\}\right\} \oplus \\ \left\{\{1-(1-s_2)^{\omega'_2}: s_2 \in \mathrm{H}_2^+\}, \{-(-s_2)^{\omega'_2}: s_2 \in \mathrm{H}_2^-\}\right\} \end{matrix}$$

$$= \left\{\begin{matrix} \left\{\begin{matrix} 1-(1-s_1)^{\acute{\omega}_1} + 1-(1-s_2)^{\acute{\omega}_2} - \\ \left(\left(1-(1-s_1)^{\acute{\omega}_1}\right)\left(1-(1-s_2)^{\acute{\omega}_2}\right)\right): \\ s_1 \in \mathrm{H}_1^+, s_2 \in \mathrm{H}_2^+ \end{matrix}\right\}, \\ \left\{\begin{matrix} -\left(-(-s_1)^{\acute{\omega}_1}\right)\left(-(-s_2)^{\acute{\omega}_2}\right): \\ s_1 \in \mathrm{H}_1^-, s_2 \in \mathrm{H}_2^- \end{matrix}\right\} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \left\{\begin{matrix} 1-(1-s_1)^{\acute{\omega}_1} + 1-(1-s_2)^{\acute{\omega}_2} - 1 + \\ (1-s_1)^{\acute{\omega}_1} + (1-s_2)^{\acute{\omega}_2} - \\ (1-s_1)^{\acute{\omega}_1}(1-s_2)^{\acute{\omega}_2}: s_1 \in \mathrm{H}_1^+, s_2 \in \mathrm{H}_2^+ \end{matrix}\right\}, \\ \left\{\begin{matrix} -(-s_1)^{\acute{\omega}_1}(-s_2)^{\acute{\omega}_2}: \\ s_1 \in \mathrm{H}_1^-, s_2 \in \mathrm{H}_2^- \end{matrix}\right\} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \{1-(1-s_1)^{\acute{\omega}_1}(1-s_2)^{\acute{\omega}_2}: s_1 \in \mathrm{H}_1^+, s_2 \in \mathrm{H}_2^+\}, \\ \{-(-s_1)^{\omega'_1}(-s_2)^{\omega'_2}: s_1 \in \mathrm{H}_1^-, s_2 \in \mathrm{H}_2^-\} \end{matrix}\right\}$$

Now let the result holds for $n = k$ i.e.

$BPVHFWA(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_k) =$
$\left\{\begin{matrix} \{1-\prod_{i=1}^{k}(1-s_i)^{\acute{\omega}_i}: s_i \in \mathrm{H}_i^+\}, \\ \{-\prod_{i=1}^{k}(-s_i)^{\omega'_i}: s_i \in \mathrm{H}_i^-\} \end{matrix}\right\}.$

Then when $n = k + 1$

$$BPVHFWA(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_{k+1}) = \bigoplus_{i=1}^{K+1} (\omega'_i \mathrm{H}_i)$$

$$= \bigoplus_{i=1}^{K} (\omega'_i \mathrm{H}_i) \oplus (\omega'_{K+1} \mathrm{H}_{K+1}).$$

$$= \left\{\begin{matrix} \left\{1-\prod_{i=1}^{k}(1-s_i)^{\acute{\omega}_i}: s_i \in \mathrm{H}_i^+\right\}, \\ \left\{-\prod_{i=1}^{k}(-s_i)^{\omega'_i}: s_i \in \mathrm{H}_i^-\right\} \end{matrix}\right\} \oplus \left\{\begin{matrix} \{1-(1-s_{k+1})^{\acute{\omega}_{k+1}}: s_{k+1} \in \mathrm{H}_{k+1}^+\}, \\ \{-(-s_{k+1})^{\omega'_{k+1}}: s_{k+1} \in \mathrm{H}_{k+1}^-\} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \left\{\begin{matrix} 1-\prod_{i=1}^{k}(1-s_i)^{\omega'_i} + 1-(1-s_{k+1})^{\omega'_{k+1}} - \\ \left(1-\prod_{i=1}^{k}(1-s_i)^{\acute{\omega}_i}\right)\left(1-(1-s_{k+1})^{\acute{\omega}_{k+1}}\right): \\ s_i \in \mathrm{H}_i^+, s_{k+1} \in \mathrm{H}_{k+1}^+ \end{matrix}\right\}, \\ \left\{\begin{matrix} -\left(-\prod_{i=1}^{k}(-s_i)^{\omega'_i}\right)\left(-(-s_{k+1})^{\omega'_{k+1}}\right): \\ s_i \in \mathrm{H}_i^-, s_{k+1} \in \mathrm{H}_{k+1}^- \end{matrix}\right\} \end{matrix}\right\}$$

$$= \left\{ \begin{array}{c} \left\{1 - \prod_{i=1}^{k+1} (1 - s_i)^{\acute{\omega}_i} : s_i \in \mathrm{H}_i^+ \right\}, \\ \left\{ - \prod_{i=1}^{k+1} (-s_i)^{\omega'_i} : s_i \in \mathrm{H}_i^- \right\} \end{array} \right\}$$

The result is true for $n = k + 1$. Hence the result holds true$\forall\ n \geq 2$. This completes the proof of theorem.

Definition 5:

For BPVHFEs$H_i (i = 1,2,3,4 \ldots n)$, the BPVHFWG operator is defined as:

$$BPVHFWG(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n) = \bigotimes_{i=1}^{n} (\mathrm{H}_i)^{\omega'_i}$$

Theorem 3:

The aggregated value of BPVHFEs$H_i (i = 1,2,3 \ldots n)$ determined by using BPVHFWG operator is a BPVHFE and

$BPVHFWG(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n)$

$$= \left\{ \begin{array}{c} \left\{ \prod_{i=1}^{n} (s_i)^{\acute{\omega}_i} : s_i \in \mathrm{H}_i^+ \right\}, \\ \left\{ -1 - \prod_{i=1}^{n} \left(-\left(-(-1 - s_i)\right)^{\omega'_i} \right) : s_i \in \mathrm{H}_i^- \right\} \end{array} \right\}$$

The proof is similar to the previous theorem.

Definition 6:

For BPVHFEs$H_i (i = 1,2,3,4 \ldots n)$ the generalized aggregation operators are defined as:

1. A GBPVHFWA operator is defined as:

$$GBPVHFWA_{\rho^{\cdot}}(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n) = \left(\bigoplus_{i=1}^{n} \left(\omega'_i \mathrm{H}_i^{\rho^{\cdot}} \right) \right)^{\frac{1}{\rho^{\cdot}}}$$

$$with \rho^{\cdot} > 0$$

2. A GBPVHFWG operator is defined as:

$$GBPVHFWG_{\rho^{\cdot}}(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n) = \frac{1}{\rho^{\cdot}} \left(\bigoplus_{i=1}^{n} (\rho^{\cdot} \mathrm{H}_i)^{\omega'_i} \right)$$

$$with \rho^{\cdot} > 0$$

When $\rho^{\cdot} = 1$ then GBPVHFWA operator becomes BPVHFWA operator and GBPVHFWG operator becomes BPVHFWG operator. Moreover, the above definitions can also be defined as

$$GBPVHFWA_{\rho^{\cdot}}(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n) = \left(\bigoplus_{i=1}^{n} \left(\omega'_i \mathrm{H}_i^{\rho^{\cdot}} \right) \right)^{\frac{1}{\rho^{\cdot}}}$$

$$= \left\{ \begin{array}{c} \left\{ \left(1 - \prod_{i=1}^{n} \left(1 - s_i^{\dot{\rho}}\right)^{\acute{\omega}_i} \right)^{\frac{1}{\dot{\rho}}} : s_i \in \mathrm{H}_i^+ \right\}, \\ \left\{ -1 - \left(\prod_{i=1}^{n} - \left(- \left(-1 \left(-((-s_i)^{\rho^{\cdot}})^{\omega'_i} \right) \right) \right) \right)^{\frac{1}{\rho^{\cdot}}} : s_i \in \mathrm{H}_i^- \right\} \end{array} \right\}$$

Similarly

$$GBPVHFWG_{\rho^{\cdot}}(\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_n) = \frac{1}{\rho^{\cdot}} \left(\bigoplus_{i=1}^{n} (\rho^{\cdot} \mathrm{H}_i)^{\omega'_i} \right)$$

$$= \left\{ \begin{array}{c} \left\{ 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - s_i)^{\dot{\rho}} \right)^{\acute{\omega}_i} \right)^{\frac{1}{\dot{\rho}}} : s_i \in \mathrm{H}_i^+ \right\}, \\ \left\{ -1 - \left(\prod_{i=1}^{n} - \left(- \left(-1 - (-(-s_i)^{\rho^{\cdot}}) \right) \right)^{\omega'_i} \right)^{\frac{1}{\rho^{\cdot}}} : s_i \in \mathrm{H}_i^- \right\} \end{array} \right\}$$

Definition 7:

The score function of a BPVHFE His defined as:

$$\mathcal{S}(\mathrm{H}) = \frac{1}{\ell_{\mathrm{H}}} (\xi_{\mathrm{H}}^+ + \xi_{\mathrm{H}}^-)$$

Where ξ_{H}^+ and ξ_{H}^- denote the sum of all elements of $\mathrm{H}^+$and $\mathrm{H}^-$and ℓ_{H} denote the length of H.

Remark 1:

Length of $\mathrm{H}^+$and H^- are not necessarily equal.

Example 2:

Let $\mathrm{H} = \{\{0.1, 0.2, 0.5, 0.33\}, \{-0.3, -0.2\}\}$ then $\mathcal{S}(\mathrm{H}) = 0.0325$

For two BPVHFEs H_1 and H_2, if

$\mathcal{S}(\mathrm{H}_1) < \mathcal{S}(\mathrm{H}_2)$ implies that $\mathrm{H}_1 < H_2$.

$\mathcal{S}(\mathrm{H}_1) > \mathcal{S}(\mathrm{H}_2)$ implies that $\mathrm{H}_1 > H_2$.

$\mathcal{S}(\mathrm{H}_1) = \mathcal{S}(\mathrm{H}_2)$ implies that $\mathrm{H}_1 \sim \mathrm{H}_2$.

Example 3:

Let
$\mathrm{H}_1 = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$, $\mathrm{H}_2 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$ and
$\mathrm{H}_3 = \{\{0.9, 0.8\}, \{-0.2, -0.1\}\}$.

$\mathcal{S}(\mathrm{H}_1) = -0.1, \mathcal{S}(\mathrm{H}_2) = 0.8, \mathcal{S}(\mathrm{H}_3) = 0.7.$

Clearly as

$\mathcal{S}(\mathrm{H}_1) < \mathcal{S}(\mathrm{H}_2)$ so $\mathrm{H}_1 < H_2$ also $\mathcal{S}(\mathrm{H}_2) > \mathcal{S}(\mathrm{H}_3)$ so $\mathrm{H}_2 > H_3$

Example 4:

Let
$\mathrm{H}_1 = \{\{0.2, 0.3\}, \{-0.3, -0.2\}\}$, $\mathrm{H}_2 = \{\{0.5, 0.6\}, \{-0.5, -0.6\}\}$ then

$\mathcal{S}(\mathrm{H}_1) = 0 = \mathcal{S}(\mathrm{H}_2)$

As $\mathcal{S}(\mathrm{H}_1) = \mathcal{S}(\mathrm{H}_2)$, so H_1 is indifferent (similar) to H_2 denoted by $\mathrm{H}_1 \sim \mathrm{H}_2$.

Since, the above definition does not differentiate between two BPVHFEs when their score has same value. So, to differentiate them we define deviation degree as to differentiate between two BPVHFEs, when they have same score value but different deviation degree value.

Definition 8:

Let $\mathrm{H} = < \mathrm{H}^+, \mathrm{H}^- >$ be a BPVHFE, then the deviation degree of H is denoted and defined by:

$$\bar{\sigma}(\mathrm{H}) = \left[\frac{1}{\ell_{\mathrm{H}}} \sum_{s \in \mathrm{H}} \left(s - \mathcal{S}(\mathrm{H})\right)^2\right]^{\frac{1}{2}}$$

Here $\mathcal{S}(\mathrm{H})$denote the score BPVHFE.

Definition 9:

Let $\mathrm{H}_1, \mathrm{H}_2$ be two BPVHFEs and $\mathcal{S}(\mathrm{H}_1)$, $\mathcal{S}(\mathrm{H}_2)$ be their score functions and $\bar{\sigma}(\mathrm{H}_1)$, $\bar{\sigma}(\mathrm{H}_2)$ be their deviations degrees respectively. If $\mathcal{S}(\mathrm{H}_1) = \mathcal{S}(\mathrm{H}_2)$. Then

$\bar{\sigma}(\mathrm{H}_1) = \bar{\sigma}(\mathrm{H}_2)$ implies $\mathrm{H}_1 = \mathrm{H}_2$

$\bar{\sigma}(\mathrm{H}_1) < \bar{\sigma}(\mathrm{H}_2)$ implies $\mathrm{H}_1 > H_2$

$\bar{\sigma}(\mathrm{H}_1) > \bar{\sigma}(\mathrm{H}_2)$ implies $\mathrm{H}_1 < H_2$

Example 5:

Let
$\mathrm{H}_1 = \{\{0.2, 0.3\}, \{-0.3, -0.2\}\}$, $\mathrm{H}_2 = \{\{0.5, 0.6\}, \{-0.5, -0.6\}\}$ and
$\mathrm{H}_3 = \{\{0.9, 0.8\}, \{-0.2, -0.1\}\}$. Then

$\mathcal{S}(\mathrm{H}_3) = 0.7 > \mathcal{S}(\mathrm{H}_1) = 0$ and $\mathcal{S}(\mathrm{H}_3) = 0.7 > \mathcal{S}(\mathrm{H}_2) = 0$ so $\mathrm{H}_3 > H_1$and $\mathrm{H}_3 > H_2$.

Now as $\mathcal{S}(\mathrm{H}_1) = 0 = \mathcal{S}(\mathrm{H}_2)$ so

$\bar{\sigma}(\mathrm{H}_1) = 0.36, \bar{\sigma}(\mathrm{H}_2) = 0.78$

As$\bar{\sigma}(\mathrm{H}_1) < \bar{\sigma}(\mathrm{H}_2)$ so$\mathrm{H}_1 > H_2$ hence $\mathrm{H}_3 > \mathrm{H}_1 > \mathrm{H}_2$

2. Application

In this section, the MADM is described in the environment of BPVHFSs. In this approach the evaluation of best alternatives is achieved using the aggregation operators of BPVHFSs. If the number of alternative be n with $m$attributes and w be their weight vector provided that $w_j \in [-1,1]$, $j = 1,2,3 \dots m$ and $\sum_{j=1}^{m} \omega'_j = 1$. Remaining in the state of being anonymous, the decision makers gave their information in the form of BPVHFESs. The step by step algorithm of proposed method is explained as follows.

Step 1:

In this step each alternativeA_i has assigned some values BPVHFEs (H_{ij}) under some attributesm_j i.e. the decision matrix is formed.

Step 2:

In this step, using one of the aggregation operator, BPVHFE $\alpha_i (i = 1,2,3, \dots, n)$ can be obtained for the alternatives$A_i (i = 1,2,3, \dots, n)$.

Step 3:

Using ranking functions, we can find the rank of each $(\alpha_i), (i = 1,2,3, \dots, n)$.

Step 4:

By ordering the score values $\mathcal{S}(\alpha_i), (i = 1,2,3, \dots, n)$ we can get the best alternative $A_i (i = 1,2,3, \dots, n)$

Example 6:

A bank needed a manager in one of its branch office. The bank announced the post in a newspaper. Several candidates applied for the post and after initial screening 4 candidates were called for an interview. The governing body will have to select a person who possessed the qualities like hard working, creative, flexible, committed and disciplined. It is necessary to select the most suitable persons among these 4 persons.

Let $A = \{A_1, A_2, A_3, A_4\}$ be the set of alternatives and $X = \{m_1, m_2, m_3, m_4, m_5\}$ be the set of attributes and let $w = (0.25, 0.1, 0.13, 0.35, 0.17)^T$ be the weight vector of the attributes $X_i (i = 1,2,3,4,5)$.

Now we use the MADM method to get the most suitable candidate.

Step 1:

Avoiding any kind of influence, the decision makers, in the state of being anonymous, presented the decision matrix shown in the Table 1.

Step 2:

Now we apply GBPVHFWA operator to get BPVHFE $H_i (i = 1,2,3,4)$ for candidates$A_i (i = 1, 2, 3, 4)$

We take the 1st candidate i.e. A_1. Let $\rho^{\cdot} = 1$.

$H_1 = GBPVHFWA_1(H_{11}, H_{12}, H_{13}, H_{14}, H_{15})$

Table 1: Decision matrix.

	κ_1	κ_2	κ_3	κ_4	κ_5
A_1	$\left\{\begin{matrix}\{0.5, 0.6\},\\ \{-0.3, -0.2\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.1, 0.3\},\\ \{-0.5, -0.3\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.2, 0.3\},\\ \{-0.3, -0.2\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.1, 0.3\},\\ \{-0.1, -0.2\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.6, 0.1\},\\ \{-0.9, -0.6\}\end{matrix}\right\}$
A_2	$\left\{\begin{matrix}\{0.2, 0.3\},\\ \{-0.4, -0.2\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.6, 0.7\},\\ \{-0.4, -0.1\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.2, 0.3\},\\ \{-0.3, -0.2\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.4, 0.3\},\\ \{-0.6, -0.4\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.5, 0.3\},\\ \{-0.2, -0.5\}\end{matrix}\right\}$
A_3	$\left\{\begin{matrix}\{0.1, 0.2\},\\ \{-0.4, -0.3\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.4, 0.3\},\\ \{-0.7, -0.6\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.2, 0.3\},\\ \{-0.3, -0.2\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.2, 0.7\},\\ \{-0.3, -0.5\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.4, 0.2\},\\ \{-0.4, -0.6\}\end{matrix}\right\}$
A_4	$\left\{\begin{matrix}\{0.2, 0.4\},\\ \{-0.6, -0.5\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.3, 0.5\},\\ \{-0.5, -0.1\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.2, 0.3\},\\ \{-0.3, -0.2\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.4, 0.1\},\\ \{-0.1, -0.4\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{0.4, 0.1\},\\ \{-0.1, -0.3\}\end{matrix}\right\}$

$$= BPVHFWA\begin{pmatrix}\{\{0.5, 0.6\}, \{-0.3, -0.2\}\}, \{\{0.1, 0.3\}, \{-0.5, -0.3\}\}, \{\{0.2, 0.3\}, \{-0.3, -0.2\}\},\\ \{\{0.1, 0.3\}, \{-0.1, -0.2\}\}, \{\{0.6, 0.1\}, \{-0.9, -0.6\}\}\end{pmatrix}$$

$$\mathrm{H}_1 = \left\{\begin{matrix}\left\{\begin{matrix}0.333342, 0.234801, 0.389476, 0.299232, 0.344814, 0.247969, 0.399982, 0.311292,\\ 0.349887, 0.253792, 0.404628, 0.316624, 0.361075, 0.266633, 0.414874, 0.328385,\end{matrix}\right\},\\ \left\{\begin{matrix}-0.259074, -0.241818, -0.330206, -0.308212, -0.245772, -0.229402, -0.313251, -0.292387,\\ -0.246172, -0.229776, -0.313762, -0.292863, -0.233533, -0.217978, -0.297651, -0.277826\end{matrix}\right\}\end{matrix}\right\}$$

In the same way

$$\mathrm{H}_2 = \left\{\begin{matrix}\left\{\begin{matrix}0.3769, 0.340219, 0.342358, 0.303644, 0.387623, 0.351573, 0.353676, 0.315628,\\ 0.39457, 0.358929, 0.361008, 0.323392, 0.404989, 0.369962, 0.372004, 0.335036\end{matrix}\right\},\\ \left\{\begin{matrix}-0.394707, -0.461237, -0.342486, -0.400214, -0.374441, -0.437555, -0.324901, -0.379665,\\ -0.343612, -0.401531, -0.298151, -0.348407, -0.325969, -0.380914, -0.282843, -0.330518\end{matrix}\right\}\end{matrix}\right\}$$

$$H_3 = \left\{\begin{matrix}\left\{\begin{matrix}0.237667, 0.199458, 0.459176, 0.43207, 0.250786, 0.213235, 0.468484, 0.433172,\\ 0.441843, 0.187022, 0.450775, 0.423247, 0.239148, 0.225825, 0.201013, 0.460227\end{matrix}\right\},\\ \left\{\begin{matrix}-0.368462, -0.394756, -0.440596, -0.472037, -0.349544, -0.374487, -0.417973, -0.4478,\\ -0.362826, -0.388717, -0.433856, -0.464816, -0.344197, -0.368759, -0.411579, -0.44095\end{matrix}\right\}\end{matrix}\right\}$$

$$H_4 = \left\{\begin{matrix}\left\{\begin{matrix}0.320292, 0.271787, 0.216652, 0.160752, 0.331989, 0.284319, 0.230133, 0.175195,\\ 0.342781, 0.295882, 0.242571, 0.188521, 0.354092, 0.308, 0.255606, 0.202486\end{matrix}\right\},\\ \left\{\begin{matrix}-0.212061, -0.255606, -0.344494, -0.415233, -0.201173, -0.242482, -0.326806, -0.393913,\\ -0.180536, -0.217608, -0.293281, -0.353505, -0.171266, -0.206435, 0.278223, -0.335354\end{matrix}\right\}\end{matrix}\right\}$$

Table 2: Score values of data determined using aggregation operators.

$GBPVHFWA_{\dot{\rho}}$	A_1	A_2	A_3	A_4
$GBPVHFWA_1$	0.057945	−0.008478	−0.072388	−0.015432
$GBPVHFWA_5$	−0.27503	−0.308275	−0.432227	−0.437984
$GBPVHFWA_{10}$	0.213082	0.181159	−0.03303	−0.03175
$GBPVHFWA_{15}$	0.396762	0.417291	0.219835	0.189746
$GBPVHFWA_{20}$	0.464247	0.518946	0.35378	0.300758
$GBPVHFWA_{30}$	0.508244	0.588962	0.429868	0.350934

Step 3:

Now we calculate the values of $GBPVHFWA_{\dot{\rho}}$ for $\dot{\rho} = 5, 10, 15, 20, 30$and then wecalculate the score (accuracy value) of eachH_i i.e. $S(H_i), (i = 1,2,3,4)$. The score corresponding to alternatives are shown in the Table 2.

Step 4:

Finally, we made a comparison among the score values of each alternative to choose the best we need. The comparative analysis of score values for different values of $\dot{\rho}$is demonstrated follows:

For $\dot{\rho} = 1$, we have $A_1 > A_2 > A_4 > A_3$

For $\dot{\rho} = 5$, we have $A_1 > A_2 > A_3 > A_4$

For $\dot{\rho} = 10$, we have $A_1 > A_2 > A_4 > A_3$

For $\dot{\rho} = 15$, we have $A_2 > A_1 > A_3 > A_4$

For $\dot{\rho} = 20$, we have $A_2 > A_1 > A_3 > A_4$

For $\dot{\rho} = 30$, we have$A_2 > A_1 > A_3 > A_4$

Our analysis shows that candidate $A_1$is more suitable when $\rho^{\cdot}$ less than 10 but as we increase the value of $\rho^{\cdot}$ we came to know that candidate A_2 becomes more prominent. The DMs may choose the value of $\rho^{\cdot}$ by their own consent. If we want the best result, then we should keep varying the value of $\rho^{\cdot}$ so that any kind of ambiguity or uncertainty can be removed.

The comparison of score values may alter if instead of $GBPVHFWA$ operator we simply use $GBPVHFWG$ operator.

Conclusion

In our treatise, the concept of BPVHFS is proposed. Some operations of BPVHFSs are discussed and their properties are studied. Based on defined operations some aggregation operators are conferred. The aggregation operations are further applied to solve a decision-making problem. The structure of BPVHFSs is basically a mixture of HFSs and BVFSs and can be applied to many situations like medical diagnosis, clustering, digital image processing, communications and networking problems etc.

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