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Paramedial and Bol* Abel-Grassmann Groupoids

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ABSTRACT

We introduce a new class of groupoids which we name Bol-groupoids. This in turn gives another new class of AG-groupoids that we call Bol*-AG-groupoids. We also enlist two new classes of AG-groupoids namely paramedial AG-groupoids and AG-groupoid-semigroups. The latter is also an interesting new class of semigroups. We discuss some basic properties of these newly introduced classes and provide various finite examples to show their existence.*

1. Introduction

We prove the existence of a new class of groupoids which we call Bol*-groupoids. A groupoid is called Bol*-groupoid if it satisfies the identity $(ab\cdot c)d = a\cdot(bc\cdot d)$. We call it Bol*-groupoid because by taking $d=b$ it becomes the famous right Bol identity. By a construction we prove that Bol*-groupoids are obtainable from semigroups through endomorphisms of semigroups. However, due to the unavailability of all the endomorphisms of semigroups we are unable to confirm that we get all Bol*-groupoids from semigroups. We introduce a new class of semigroups and Abel-Grasmann groupoids (or shortly AG-groupoids) which we call AG-groupoid semigroups, that is, a groupoid which satisfies left invertive law as well as the associative law. This class contains all commutative semigroups. However, we study non-commutative AG-groupoid semigroups because they are more interesting. This class has quite different properties from the already known famous subclasses of semigroups. It should be noted that recently we have discovered some more new classes of AG-groupoids [10, 11]. Section 2 of this article contains the construction of Bol*-groupoid from semigroups, while Section 3 consists of paramedial AG-groupoids and Bol*-AG-groupoids. The important results of this section are that, Bol*-AG-groupoid is the generalization of AG**-groupoid and a special kind of paramedial AG-groupoid and that the in-existence of non-associative paramedial AG-3-band. We emphasis on non-commutative AG-groupoid semigroup in Section 4 that it does not lie in the known classes of semigroups and provides an example of such semigroup in which the product of idempotents is always an idempotent. It cannot contain left identity as well as right identity and always satisfies the paramedial property.

1.1 Preliminaries

A groupoid is called AG-groupoid if it satisfies the left invertive law: $(ab)c=(cb)a$ [8]. An AG-groupoid S which satisfy $(ab)c=b(ac)$, for all $a,b,c\in S$, is called AG*-groupoid. An AG**-groupoid is an AG-groupoid satisfying the identity $a(bc)=b(ac)$. S with left identity is called AG-monoid. Every AG-monoid is AG**-groupoid. An AG-groupoid S always satisfies the medial law: $(ab)(cd)=(ac)(bd)$ [5, Lemma1.1 (i)], while an AG-monoid satisfies paramedial law: $(ab)(cd)=(db)(ca)$ [5, Lemma 1.2 (ii)]. An AG-groupoid is called paramedial AG-groupoid if it satisfies paramedial law. Note that in [5] the name right modular groupoid is used for AG-groupoid. An element a of an AG-groupoid S is called idempotent if $a^2=a$. An AG-groupoid S is called idempotent or AG-2-band or simply AG-band [2] if its every element is idempotent. S is called AG-3-band if its every element satisfies $a(aa)=(aa)a=a$ [4]. AG-groupoids have a variety of applications in various fields of mathematics. Some applications of AG-groupoids in flocks theory are given in [1] and some of its applications in geometry and in finite mathematics have been investigated in [6, 12]. For further studies on AG-

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groupoids we refer the reader to [3, 6] while, for the semigroup concepts we suggest [9]. The following table contains various AG-groupoids with their defining identities that will be used in the rest of this article.

Table 1: Identities of various AG-groupoids

AG-groupoid	Defining identities
Left nuclear square AG-groupoid (LNS)	$a^2(bc) = (a^2b)c$
Middle nuclear square AG-groupoid (MNS)	$(ab^2)c = a(b^2c)$
Right nuclear square AG-groupoid (RNS)	$(ab)c^2 = a(bc^2)$
Unipotent AG-groupoid	$a^2 = b^2$
T^1-AG-groupoid	$ab = cd \Rightarrow ba = dc$
T^2-AG-groupoid	$ab = cd \Rightarrow ca = bd$
Transitively commutative AG-groupoid	$ab = ba$ & $bc = cb \Rightarrow ac = ca$
Quasi-cancellative AG-groupoid	$x^2 = xy$ & $y^2 = yx \Rightarrow x = y$ $x^2 = yx$ & $y^2 = xy \Rightarrow x = y$

Table 2 presents the enumeration of new subclasses of AG-groupoids. Note that except for non-commutative AG-groupiod semigroup, only the numbers of non-associative AG-groupoids are shown.

Table 2: Classification and enumeration results for new subclasses of AG-groupoids of orders 3-6

Order	3	4	5	6
Total	20	331	31913	40104513
Paramedial AG-groupoids	8	264	31006	39963244
Bol*-AG-groupoids	4	58	2706	1357494
Non-commutative AG - groupiod semigroup	0	4	121	5367

2. Construction of Bol*-groupoid from Semigroups

In the following we prove that from a given Bol*-groupoid and each fixed element p of S we can obtain a semigroup by defining a new operation $*$ on S by

$$x * y = (xp)y \quad \forall \quad x, y \in S$$

Proposition 1: Let $(S,*)$ be a Bol*-groupoid. Define

$$* : S \times S \rightarrow S \text{ by } (x, y) = xp * y$$

where p is any fixed element of S. Then $(S,*)$ is a semigroup.

Proof: Let $x, y, z \in S$. The set $(S,*)$ is closed since $xp \cdot y$ is an element of S. Thus we only check the property of associativity as follows :

$$(x * y) * z = (((xp)y)p)z$$

$$= (xp)((yp)z) \qquad \text{by Bol}^*\text{-groupoid}$$
$$= x * (y * z).$$

Hence $(S,*)$ is a semigroup.

*Definition 2***:** An AG-groupoid S satisfying the identity

$$a(bc \cdot d) = (ab \cdot c)d \quad a, b, c, d \in S \tag{2.1}$$

is called Bol*-AG-groupoid. The following example shows that a Bol*-groupoid is not always a Bol*-AG-groupoid.

Example 3: A Bol*-groupoid that is not Bol*-AG-groupoid.

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	3	4

In order to illustrate Proposition 1, we provide the following example.

Example 4 : Given the above Bol*-groupoid $(S,\cdot)$ as in Example 3 then using Proposition 1 with fixed element $p = 1$, we get the semigroup $(S,*)$ as in the following table.

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

In the following we prove that from a semigroup we can obtain a Bol*-groupoid through suitable automorphisms.

*Theorem 5***:** Let $(S,\circ)$ be a semigroup. Let $\alpha, \beta \in End(S)$ such that

$$\alpha^3 = \alpha,\ \alpha^2\beta = \beta\alpha^2,\ \beta\alpha\beta = \alpha\beta, \beta^2 = \beta$$

Define $x \cdot y = \alpha(x) \circ \beta(y) \ \forall \ x, y \in S$ then $(S,\cdot)$ is a Bol*-groupoid.

Proof : Let $a, b, c, d \in S$. Then by definition, we have

$$\begin{aligned} ((a \cdot b) \cdot c)) \cdot d &= \alpha(\alpha(\alpha(a) \circ \beta(b)) \circ \beta(c)) \circ \beta(d) \\ &= \alpha^3(a) \circ \alpha^2\beta(b)) \circ \alpha\beta(c)) \circ \beta(d) \end{aligned} \tag{2.2}$$

On the other hand, by definition and by using associativity, we have

$$a\cdot((b\cdot c)\cdot d)=\alpha(a)\circ((\beta\alpha^2(b)\circ\beta\alpha\beta(c))\circ\beta^2(d) \\ =((\alpha(a)\circ\beta\alpha^2(b))\circ\beta\alpha\beta(c))\circ\beta(d) \quad (2.3)$$

From (2.2) and (2.3), we can say that $(S,\cdot)$ is a Bol*-groupoid if

$$\alpha^3=\alpha,\ \alpha^2\beta=\beta\alpha^2,\ \ \beta\alpha\beta=\alpha\beta,\beta^2=\beta\ .$$

Remark 6 **:** Note that if β is identity endomorphism I then $(S,\cdot)$ is a Bol*-groupoid if $\alpha^3=\alpha$ and if α is the identity endomorphism I then $(S,\cdot)$ is a Bol*-groupoid if $\beta^2=\beta.$

Unfortunately, since all the endomorphisms are not available so we cannot confirm that whether this construction will give us all the Bol*-groupoids from a semigroup or not. As a special case of this in [6] all Bol*-quasigroups have been obtained from groups through involutive automorphisms because groups of large orders are available along with their automorphisms. The method has been implemented in AGGROUPOIDS [6], which gives all Bol*-quasigroups of order n. We have also implemented the method for Bol*-groupoids through involutive automorphisms of semigroups which gives many examples of Bol*-groupoids up to order 8 but obviously not all.

2. Paramedial AG-groupoids and Bol*-AG-Groupoids

In this section we consider the class of paramedial AG-groupoids and Bol*-AG-groupoids. We investigate some basic properties of these classes and give their relations with other classes of AG-groupoids. Some immediate observations from the definition of paramedial AG-groupoid S are:

i. Square of elements commutes with each other andtherefore an AG-band which is also paramedial AG-groupoid must be commutative semigroup;

ii. The identity $(ab)^2=(ba)^2$ holds;

iii. The identity $(ab)(cd)=(dc)(ba)$ holds;

iv. Every paramedial groupoid with left identity becomes an AG-groupoid.

Example 7. A paramedial AG-groupoid of order 4.

·	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	1	4

Example 8. A Bol*-groupoid of order 3.

·	1	2	3
1	1	1	1
2	1	1	1
3	1	2	2

Theorem 9: Non-associative paramedial AG-3-band does not exist.

Proof : Let S be a paramedial AG-3-band. Let $a,b\in S$. Now,

$$ab=a^3b^3=(ab)(a^2b^2)=(b^2a^2)(ba)=b^3a^3=ba$$

Thus S is commutative. Further, it is an easy exercise to show that S is associative, thus S is commutative semigroup. Hence the theorem is proved.

An element a of an AG-groupoid S is called left nuclear square [6, 13] if $a^2(bc)=(a^2b)c,\ \forall\ a,b,c\in S.$

Theorem 10: Every paramedial AG-groupoid is left nuclear square.

Proof : Let be a paramedial-AG-groupoid, $a,b,c\in S$. Then

$$\begin{aligned} a^2(bc)&=(aa)(bc)\\ &=ca\cdot ba \quad \text{by parmedial law}\\ &=cb\cdot a^2 \quad \text{by medial law}\\ &=(a^2b)c \quad \text{by left invertive law}\end{aligned}$$

Hence, paramedial-AG-groupoid is left nuclear square.

Proposition 11 : Every AG**-groupoid S is a Bol*-AG-groupoid.

Proof : Let $a,b,c,d\in S.$ Then

$$\begin{aligned}(ab\cdot c)d&=(dc)(ab) \quad \text{by left invertive law}\\ &=a(dc\cdot b) \quad \text{by AG}**-\text{groupoid}\\ &=a(bc\cdot d) \quad \text{by left invertive law}\end{aligned}$$

Thus S is an Bol*-AG-groupoid.

Proposition 12: Every Bol*-AG-groupoid is paramedial AG-groupoid.

Proof: Let $a,b,c,d\in S.$ Then

$$\begin{aligned} ab\cdot cd &= (cd\cdot b)a \quad \text{by left invertive law}\\ &= (bd\cdot c)a \quad \text{by left invertive law}\\ &= b(dc\cdot a) \quad \text{by (2.1)}\\ &= b(ac\cdot d) \quad \text{by left invertive law}\\ &= (ba\cdot c)d \quad \text{by (2.1)}\end{aligned}$$

$= (ca \cdot b)d$ by left invertive law

$= db \cdot ca$ by left invertive law

Hence S is paramedial AG-groupoid.

By the above two lemmas, Bol*-AG-groupoid is the generalization of AG**-groupoid and a special kind of paramedial AG-groupoid.

Proposition 13: Every Bol*-groupoid with right identity e is a semigroup.

Proof: Take $d = e$ in (2.1)

As Example 8 shows, a Bol*-groupoid not necessarily has left identity, but it can have it without becoming semigroup as the following example shows:

Example 14. A non-associative Bol*-groupoid of order 5.

·	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	4
4	1	1	1	1	3
5	1	2	3	4	5

Lemma 15: Let S be a Bol*-groupoid having a left identity. Then for all $a, b, c \in S$.

(i) $a(bc) = (ae \cdot b)c,$

(ii) $(ab \cdot e)\ c = a(be \cdot c).$

Proof: (i) Taking $b = e$ in (2.1) and re-naming

(ii) Taking $c = e$ in (2.1) and re-naming.

A left identity is left cancellative by its definition but not necessarily right cancellative as Example 14 shows, but if the left identity is also right cancellative as in the following example, then it can have more properties as in the lemma following the example.

Example 16: Bol*-groupoid of order 4 with cancellative left identity 1.

·	1	2	3	4
1	1	2	3	4
2	2	1	3	4
3	4	3	3	4
4	3	4	3	4

Lemma 17: Let S be a Bol*-groupoid having left identity e such that e is cancellative. Then,

(i) $xe \cdot e = x \ \forall \ x \in S,$ where e is left identity in S;

(ii) $(ae \cdot be)e = ab \ \forall \ a, \ b \in S;$

(iii) $(ab)e = (ae)(be) \ \forall \ a, \ b \in S.$

Proof:

i. Taking $b = c = e$ in Lemma 15 (i) and then using right cancellativity of e.

ii. Taking $b = be$ and $c = e$ in Lemma 15 (i) and then using (i).

iii. Taking $a = ae$ and $b = be$ in Lemma 15 (ii) and then using (i).

Remark 18: In Example 16, 1 and 2 are both cancellative so here one may guess that a right cancellative might be cancellative as in the case of AG-groupoids [3]. The following example shows that this is not so.

Example 19: A Bol*-groupoid of order 4.

·	1	2	3	4
1	1	2	1	1
2	2	1	2	2
3	1	2	4	4
4	1	2	3	3

Here 4 is right cancellative but not left cancellative.

Theorem 20: An AG-band S is paramedial AG-groupoid if any of the following hold:

i. S is left nuclear square AG groupoid;

ii. S is right nuclear square AG groupoid ;

iii. S is middle nuclear square AG groupoid.

Proof: Let S be an AG-band. Then for all $a, b, c \in S$;

(i) Let S be a left nuclear square AG-groupoid then,

$ab \cdot cd = ac \cdot bd$ by medial law

$= (ac)b \cdot d$ S is LNS & is idempotent

$= (bc \cdot a)d$ by left invertive law

$= da \cdot bc$ by left invertive law

$= (da \cdot b)c$ S is LNS & is idempotent

$= cb \cdot da$ by left invertive law

$= (cb \cdot d)a$ S is LNS & is idempotent

$= ad \cdot cb$ by left invertive law

$= ac \cdot db$ by medial law

$= (db \cdot c)a$ by left invertive law

$= db \cdot ca$ S is LNS

(ii) Let S be a middle nuclear square AG-groupoid, then for all $a, b, c, d \in S$;

$ab \cdot cd = (ab \cdot c)d$ S is MNS & is idempotent

$= dc \cdot ab$ by left invertive law

$= (dc \cdot a)b$ S is MNS

$= (ac \cdot d)b$ by left invertive law
$= bd \cdot ac$ by left invertive law
$= ba \cdot dc$ by medial law
$= (ba \cdot d)c$ by left invertive law
$= cd \cdot ba$ by medial law
$= cb \cdot da$ by left invertive law
$= (da \cdot b)c$ left invertive law
$= (d \cdot ab)c$ by medial law
$= (c \cdot ab)d$ by left invertive law
$= (ca \cdot b)d$ by medial law
$= db \cdot ca$ by left invertive law

Hence S is paramedial AG-groupoid.

(iii) Let S be a right nuclear square AG-groupoid, then for all $a,b,c,d \in S$;

$ab \cdot cd = a(b \cdot cd)$ S is RNS & is idempotent
$= a(bc \cdot d)$ S is RNS & is idempotent
$= a(dc \cdot b)$ by left invertive law
$= (a \cdot dc)b$ S is RNS & is idempotent
$= (ad \cdot c)b$ S is RNS & is idempotent
$= (cd \cdot a)b$ by left invertive law
$= ba \cdot cd$ by left invertive law
$= (ba \cdot c)d$ S is RNS & is idempotent
$= (ca \cdot b)d$ by left invertive law
$= db \cdot ca$ by left invertive law

Hence S is paramedial AG-groupoid.

Theorem 21: Cancellative Bol* is transitively commutative AG-groupoid.

Proof : Let S be a cancellative Bol* AG-groupoid. Let $xy = yx$ and $yz = zy$, we show using the assumption and the left invertive law that $xz = zx$. Since

$$\begin{aligned} y(xz \cdot y) &= (yx \cdot z)y \\ &= yz \cdot yx = y(zy \cdot x) \\ &= y(xy \cdot z) = y(yx \cdot z) \\ &= y(zx \cdot y). \end{aligned}$$

This gives by cancellativity $xz \cdot y = zx \cdot y$ and thus $xz = zx$. Hence S is transitively commutative. Since every transitively commutative AG-groupoid is quasi-cancellative, the following corollary is obvious.

Corollary 22: Every cancellative Bol* is quasi-cancellative AG-groupoid.

The converse of Proposition 11 is not valid as the following example shows; however, we have the following Theorem 24.

Example 23: A Bol* AG-groupoid of order 4 which is not AG**

·	1	2	3	4
1	3	3	3	4
2	3	3	4	4
3	4	4	4	4
4	4	4	4	4

Theorem 24: Every cancellative Bol* is AG**.

Proof : Let S be a cancellative Bol* AG-groupoid and let a,b,c,d be elements of S. Then

$$\begin{aligned} &(ab \cdot c)d = dc \cdot ab \Rightarrow d(ab \cdot c) = ab \cdot dc \\ &\Rightarrow (da \cdot b)c = ab \cdot dc = (dc \cdot b)a = (bc \cdot d)a \\ &= ad \cdot bc = cd \cdot ba = (ba \cdot d)c \\ &\Rightarrow da \cdot b = ba \cdot d \Rightarrow b \cdot da = d \cdot ba, \textit{ by cancellativity of } c \end{aligned}$$

Hence S is AG**.

Corollary 25: AG-monoid is T^1.

Proof : Since every AG-monoind is AG** which is T^1 by [11].

Theorem 26: [6] Every T^1 is Bol* AG-groupoid.

Corollary 27: Every AG-monoid is Bol* AG-groupoid.

Proof : By Corollary 25 and Theorem 26.

The converse of Theorem 26 in not true in general as:

Example 28: A Bol*AG-groupoid of order 3 that is not T^1.

·	1	2	3
1	2	2	3
2	3	3	3
3	3	3	3

However, we have the following:

Theorem 29: A Bol* AG-groupoid with a right cancellative element is T^1.

Proof : Let x be a right cancellative element of a Bol* AG-groupoid S. Let $ab = cd$. Then by the assumption, left invertive law, medial law;

$$\begin{aligned} (ba \cdot x)x &= b(ax \cdot x) = b(xx \cdot a) = (bx \cdot x)a \\ &= ax \cdot bx = ab \cdot x^2 = cd \cdot x^2 = cx \cdot dx \\ &= (dx \cdot x)c = d(xx \cdot c) = d(cx \cdot x) = (dc \cdot x)x \\ &\Rightarrow ba \cdot x = dc \cdot x \\ &\Rightarrow ba = dc. \end{aligned}$$

Theorem 30: Every unipotent AG-groupoid is left nuclear square.

Proof: Let S be an unipotent AG-groupoid, and let a, b, c are elements of S. Then by definition of unipotent AG-groupoid, and left invertive law,

$$a^2 \cdot bc = c^2(bc) = cc \cdot bc = (bc \cdot c)c = (cc \cdot b)c = (c^2b)c = (a^2b)c.$$

Proposition 31 **:** [14] Every unipotent AG-groupoid has a unique idempotent element.

Theorem 32: A unipotent AG-groupoid with a right cancellative element is paramedial.

Proof : Let S be an unipotent cancellative AG-groupoid, and let a, b, c, d are elements of S. Then by Theorem 30, Proposition 31 and using other relevant definitions, left invertive and medial laws we have,

$$\begin{aligned}(ad \cdot cd)x &= (x \cdot cd)(ab) = (x^2 \cdot cd)(ab) \\ &= (x^2c \cdot d)ab = (x^2c \cdot a)(db) \\ &= (ac \cdot x^2)(db) = (x^2c \cdot a)(db) \\ &= (x^2 \cdot ca)(db) = (db \cdot ca)x^2 \\ &= (db \cdot ca)x \\ \Rightarrow ab \cdot cd &= bd \cdot ca.\end{aligned}$$

Hence S is parmedial.

3. AG-groupoid Semigroups

This class can be considered as a subclass of both semigroups and AG-groupoids. Further, this class trivially contains all commutative semigroups because they satisfy both associative law and invertive law. Also, this is easy to see that this class is contained in the class of Bol*-AG-groupoids. The interesting subclass of this is the non-commutative AG-groupoid semigroups. Thus, this class lies between commutative semigroups and Bol*-AG-groupoids. This new class of semigroups is different from the other well-known subclasses of semigroups altogether as we will prove. So we consider only the non-trivial case, that is, non-commutative AG-groupoid semigroups. The existence of this class has been shown in Example 33 (i). We emphasis that this class is very interesting and useful as every member of this class enjoy at the same time the characteristics of semigroups as well as AG-groupoids and thus can produce many new results. Also, since the structure of semigroups is very well-known and well-studied at one hand and, on the other hand, the structure of AG-groupoids is relatively a new structure, and recently researchers have taken an interest of bringing different notions from semigroups into AG-groupoids. So this class can be used as a criterion for those concepts. The new notion should be defined in such a way that, when those new notions are applied to this class should not conflict with each other rather those should coincide when come to this class. So this class can be used as a tool for the correctness of such definitions. The previously produced such work and the upcoming both should be checked by researchers for the justification of new definitions. We discuss here some basic and interesting results in this section. We also give some conjectures whose counter examples do not exist at least up to order 6 and suggest the detailed study of this class as a future work.

Example 33: (i) A non-commutative AG-groupoid semigroup of order 4 and

·	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	1	2

(ii). An AG-monoid of order 5 with left identity 3

·	1	2	3	4	5
1	1	1	1	1	1
2	1	1	4	1	2
3	1	2	3	4	5
4	1	1	2	1	4
5	1	4	5	2	3

Recall that an AG-groupoid with right identity is a commutative monoid, which we do not consider here, but can contain left identity as in Example 33 (ii), 3 is left identity and hence is an AG-monoid. Also a semigroup can contain both left identity and right identity, but the class we are considering can contain neither as the following simple result shows.

Theorem 34: An AG-groupoid semigroup with left identity becomes commutative monoid.

Proof: Let S be an AG-groupoid semigroup with left identity e and let x be an arbitrary element in S. Then by left invertive law and associativity.

$$x = ex = (ee)x = (xe)e = x(ee) = xe.$$

Thus e is also the right identity and hence S is a commutative monoid.

Recall that a band in semigroups is called an AG-band in AG-groupoids. The following theorem shows that a band or AG-band does not exist for AG-groupoid semigroups.

Theorem 35: An AG-groupoid semigroup S is paramedial AG-groupoid.

Proof: Let $a, b, c, d \in S$. Then by repeated use of associative, invertive and medial laws, we have

$$\begin{aligned} ab \cdot cd &= (ab \cdot c)d = dc \cdot ab \\ &= (dc \cdot a)b = ba \cdot dc \\ &= bd \cdot ac = (bd \cdot a)c \\ &= ca \cdot bd = (ca \cdot b)d \\ &= db \cdot ca. \end{aligned}$$

Thus S is paramedial AG-groupoid.

Corollary 36: An AG-groupoid semigroup is a commutative semigroup if it is an AG-band.

Corollary 37: An AG-groupoid semigroup S cannot be a rectangular semigroup.

An AG-groupoid S is called dual if it also satisfies the right invertive law: $a(bc) = c(ba),\ \forall\ a, b, c \in S.$

Theorem 38: Every AG-groupoid semigroup is dual.

Proof. Let S be an AG-groupoid semigroup, and let $a, b, c \in S$ then,

$$\begin{aligned} a \cdot bc &= ab \cdot c && \text{by associativity} \\ &= cb \cdot a && \text{by left invertive law} \\ &= c \cdot ba && \text{by associativity} \end{aligned}$$

Hence S is dual.

The medial property of AG-groupoids shows that this class is closed under idempotents.

Theorem 39: An AG-groupoid semigroup S is closed under idempotents.

Proof: Let $a, b \in S$ such that $a^2 = a,\ b^2 = b.$ Then $ab = a^2b^2 = (ab)^2$. Thus S is closed under idempotents.

From the above theorem this also follows that non-idempotent elements of an AG-groupoid cannot be expressed as the product of idempotent elements. Thus

Corollary 40: An AG-groupoid semigroup S cannot be idempotent generated.

An element b of an AG-groupoid S is called the inverse of $a \in S$ if there exist $b \in S$ such that $a = aba$ and $b = bab.$ S is called inverse if each element of S has an inverse.

Theorem 41: An AG-groupoid semigroup S is a commutative semigroup if S is an inverse semigroup.

Proof. Let S be AG-groupoid semigroup such that S is also an inverse semigroup. Let $a, b \in S$. Then by definition $aba = a,\ bab = b.$ Now, by repeated use of associativity and invertive law, we have

$$\begin{aligned} ab &= (aba)\ (bab) = (aba)\ (ba \cdot b) \\ &= ((aba)ba))b = (b \cdot ba)(aba) \\ &= (b \cdot (ba)a)(ba) = (b \cdot a^2b)(ba) \\ &= (ba)(ab)(ba) = ba. \end{aligned}$$

Thus S is commutative semigroup.

Corollary 42: A non-commutative AG-groupoid semigroup cannot be an inverse semigroup and hence cannot be a Brandt semigroup.

Theorem 43: An AG-groupoid semigroup S is commutative semigroup if S is regular.

Proof: Let S be AG-groupoid semigroup such that S is also regular. Let $x \in S$. Since S is regular, then there exists y in S such that $xyx = x.$ Now, by repeated use of associativity and invertive law, we have

$$xyx = x \Rightarrow xy = (xy)(xy) = (xy)^2 \text{ and}$$

$$xyx = x \Rightarrow yx = (yx)(yx) = (yx)^2. \text{ Thus,}$$

$$(xy)^2 = (yx)^2 \Rightarrow (xy)(xy) = (yx)(yx)$$
$$\Rightarrow (xyx)y = (yxy)x \Rightarrow xy = yx$$

Hence, $xy = yx$. Thus S is commutative semigroup.

Corollary 44: A non-commutative AG-groupoid semigroup cannot be regular and hence cannot be a Clifford semigroup or an orthodox semigroup.

The data of non-commutative AG-groupoid semigroups up to order 6 that have been considered in [7] indicates that these are not simple, that is, these have a proper ideal so we have the following:

Conjecture: Every non-commutative AG-groupoid semigroup is non-simple.

References

[1] M. Naseeruddin, "Some studies on almost semigroups and flocks", Ph.D Thesis, The Aligarh Muslim University, India, 1970.

[2] N. Stevanovic and P.V. Protic, Abel-grassmann's bands, Quasigroups and Related Systems, vol. 11, no. 1, pp. 95–101, 2004.

[3] M. Shah, T. Shah and A. Ali, "On the cancellativity of AG-groupoids", International Mathematical Forum, vol. 6, no. 44, pp. 2187–2194, 2011.

[4] N. Stevanovic and P. V. Protic, Composition of Abel-Grassmann's 3-bands, Novi Sad J. Math. vol. 34, no. 2, pp.175 – 182, 2004.

[5] J. R. Cho, Pusan, J. Jezek and T. Kepka, Paramedial Groupoids, Czechoslovak Mathematical Journal, vol. 49, 277-290, 1996.

[6] M. Shah, "A theoretical and computational investigations of AG-groups", PhD thesis, Quaid-i-Azam University, Islamabad, 2012.

[7] A. Distler and M. Shah, V. Sorge, Enumeration of AG-groupoids, Lecture notes in computer science, vol. 6824, pp. 1-14, 2011.

[8] M. A. Kazim and M. Naseerudin, On almost semigroups, Portugaliae Mathematica, vl.36, no. 1, 41-47, 1977.

[9] J. M. Howie, "Fundamentals of Semigroup Theory", Clarendon Press, Oxford, 2003.

[10] M. Shah, I. Ahmad and A. Ali, "Discovery of new classes of AG-groupoids", Research Journal of Recent Sciences, vol. 1 no. 11, pp. 47-49, 2012.

[11] M. Shah, I. Ahmad and A. Ali, "On introduction of new classes of

[12] AG-groupoids", Research Journal of Recent Sciences, vol. 2, no. 1, p. 67-70, (2013).

[13] Amanullah, M. Rashad, I. Ahmad and M. Shah, "On Modulo AG-groupoids", Journal of Advances in Mathematics, vol. 8, no. 3, pp. 1606-1613, 2014.

[14] M. Rashad, I. Ahmad, Amanullah and M. Shah, "On relations between right alternative and nuclear square AG-groupoids", Int. Math. Forum, vol. 8, no. 5, pp. 237-243, 2013.

[15] M. Rashad, I. Ahmad, M. Shah and Z. A. Khuhro, "Left Transitive AG-groupoids", Sindh University Research Journal, Sci. Ser. vol. 46, no. 4, p. 547-552, 2014.