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CHARACTERIZATIONS OF h-HEMIREGULAR AND h-SEMISIMPLE HEMIRINGS BY INTERVAL VALUED $(\overline{\in},\overline{\in}\vee\overline{q})$-FUZZY h-IDEALS

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In this paper we define interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy *h*-subhemirings, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior *h*-ideals, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy prime *h*-ideals, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy semiprime *h*-ideals. We characterize *h*-hemiregular and *h*-semisimple hemirings by the properties of these interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy *h*-ideals.

Keywords: Interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy *h*-ideals, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior *h*-ideals, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy prime *h*-ideals, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy semiprime *h*-ideals, *h*-hemiregular, *h*-semisimple hemirings.

1. Introduction

In 1934 Vandiver [24] introduced semirings, considering as generalizations of associative rings and distributive lattices. Since then semirings have been proved a very strong tool for studying optimization theory, graph theory, coding theory, theory of automata, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, formal languages theory, analysis of computer programmes [5, 6], and the semiring $(\mathrm{R},\max,+)$ is the basis for idempotent analysis [12]. In more recent times semirings have been deeply studied, especially in relation with applications, as it provides algebraic framework for modeling [6]. A semiring with commutative addition and additive identity is called hemiring. Hemirings, appears in a natural manner, in some applications to the theory of automata, the theory of formal languages and in computer sciences [1, 7, 8, 15].

It is important to note that the ideals of semirings play an important role in the structure theory of semirings and are very useful for many purposes. However, in general, they do not coincide with the usual ring ideals. Many results in rings apparently have no analogues in semirings using only ideals. Due to this problem use of ideals in semirings is limited. In order to overcome this difficulty [9], Henriksen defined a more restricted class of ideals in semirings, called k-ideals, with the property that if a semiring *R* is a ring, then a complex in *R* is a *k*-ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals in hemirings, called now *h*-ideals, was given and investigated by Iizuka [10]. Fuzzy *h*-ideals are studied [3, 11, 14, 20, 27, 30] among others. La Torre [13] thoroughly studied *h*-ideals and k-ideals. The notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets proposed and discussed [16, 18]. In ref. [4], α,β -fuzzy ideals of hemirings are defined.

In 1965, Zadeh [28] introduced the concept of fuzzy set. Since then fuzzy sets have been extensively used in many branches of Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [19], dealing with fuzzy sets of groups. In ref. [2] J. Ahsan et al. initiated the study of fuzzy sets of semirings. The fuzzy algebraic structures play an important role in Mathematics with wide applications in theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [15, 17, 25].

In 1975 the concept of interval valued fuzzy sets was introduced by Zadeh [29], as a generalization of the notion of fuzzy sets. In ref. [14], Ma and Zhan introduced the concept of interval valued $(\in,\in\vee q)$-fuzzy h-ideals in hemirings and developed some basic results. In ref. [23] Sun et al. characterized *h*-hemiregular and h-intra-hemiregular hemirings by the properties of their interval valued fuzzy left and right h-ideals. In ref. [21, 22], author characterize different classes of hemirings by the properties their interval valued fuzzy *h*-ideals, interval valued fuzzy *h*-bi-ideals and interval valued fuzzy *h*-quasi-ideals. In this paper we define interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy *h*-subhemirings, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior *h*-

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ideals, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy prime h-ideals, interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy semiprime h-ideals. We characterize h-hemiregular and h-semisimple hemirings by the properties of these interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy h-ideals.

2. Preliminaries

A non-empty set R together with two binary operations addition "$+$" and multiplication "$\cdot$" is called semiring if $(R,+)$ and $(R,\cdot)$ are semigroups and multiplication distributes from both sides over addition, that is for all $x,y,z\in R$,

$$x.(y+z)=x.y+x.z \text{ and } (x+y).z=x.z+y.z$$

An element $0\in R$ satisfying the conditions, $0.r=r.0=0$ and $0+r=r+0=r$, for all $r\in R$, is called a zero of the semiring $(R,+,\cdot)$. An element $1\in R$ satisfying the condition, $1.r=r.1=r$, for all $r\in R$, is called identity of the semiring R. A semiring with commutative multiplication is called a commutative semiring. A semiring with commutative addition and zero element is called a hemiring. A non-empty subset H of hemiring R is called a subhemiring of R if it contains zero with $a+b\in H$ and $ab\in H$, for all $a,b\in H$. A non-empty subset I of hemiring R is called a left (right) ideal of R if I is closed under addition and $ra\in I$ $(ar\in I)$ for all $a\in I$ and $r\in R$. Furthermore I is called an ideal of R if it is both a left ideal and right ideal of R. A non-empty subset I of a hemiring R is said to be interior ideal of R, if I is closed under addition and multiplication of R and $rar\in I$, for all $a\in I$ and $r\in R$. An ideal P of a commutative hemiring R with unity, is called prime if $xy\in P\Rightarrow x\in P$ or $y\in P$, for all $x,y\in R$. An ideal S of a commutative hemiring R with unity, is called semiprime if $x^2\in S\Rightarrow x\in S$, for all $x\in R$.

A subhemiring (right ideal, left ideal, interior ideal, prime ideal, semiprime ideal) I of a hemiring R is called an h-subhemiring (right h-ideal, left h-ideal, interior h-ideal, prime h-ideal, semiprime h-ideal) if for all $x,z\in R$ and for any $a,b\in I$, from $x+a+z=b+z$ it follows $x\in I$.

A fuzzy subset λ of a universe X is a function $\lambda: X\to[0,1]$. A fuzzy subset of X of the form

$$x_t(y)=\begin{cases} t & \text{if } y=x \\ 0 & \text{if } y\neq x\end{cases}$$

is called the fuzzy point with support x and value t, where $t\in(0,1]$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set λ, written as $x_t\in\lambda$, (resp. $x_tq\lambda$), if $\lambda(x)\geq t$ (resp. $\lambda(x)+t>1$), and in this case, $x_t\in\vee q\lambda$ (resp. $x_t\in\wedge q\lambda$) means that $x_t\in\lambda$ or $x_tq\lambda$ (resp. $x_t\in\lambda$ and $x_tq\lambda$). To say that $x_t\overline{\alpha}\lambda$ means that $x_t\alpha\lambda$ does not hold, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$ [18].

For any two fuzzy subsets λ and μ of X, $\lambda\leq\mu$ means that, for all $x\in X$, $\lambda(x)\leq\mu(x)$. The symbols $\lambda\wedge\mu$, and $\lambda\vee\mu$ will mean the following fuzzy subsets of X

$$(\lambda\wedge\mu)(x)=\min\{\lambda(x),\mu(x)\}$$

$$(\lambda\vee\mu)(x)=\max\{\lambda(x),\mu(x)\}$$

The h-closure $\overline{A}$ of a non-empty subset A of a hemiring R is defined as

$$\overline{A}=\left\{\begin{matrix} x\in R \mid x+a+z=b+z \text{ for} \\ \text{some } a,b\in A, z\in R\end{matrix}\right\}.$$

3. Interval Valued Fuzzy Sets

Let Σ be the family of all closed subintervals of $[0,1]$. By an interval number $\hat{a}$ we mean an interval $[a^-,a^+]\in\Sigma$, where $0\leq a^-\leq a^+\leq 1$. The interval $[a,a]$ can be identified by the number $a\in[0,1]$. The element 0=[0, 0] is the minimal element and the element 1=[1, 1] is the maximal element of Σ, according to the partial order $(\alpha,\alpha')\leq(\beta,\beta')$ if and only if $\alpha\leq\beta,\alpha'\leq\beta'$ defined on Σ for all $(\alpha,\alpha'),(\beta,\beta')\in\Sigma$.

An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is a function $\hat{\lambda}: R\to\Sigma$. We write $\hat{\lambda}(x)=\left[\lambda^-(x),\lambda^+(x)\right]\subseteq[0,1]$, for all $x\in R$, where λ^-,λ^+ are fuzzy subsets of R such that for all $x\in R$,

$0 \le \lambda^-(x) \le \lambda^+(x) \le 1$. For simplicity we write $\hat{\lambda} = [\lambda^-, \lambda^+]$. From now to onward we will denote the set of all interval valued fuzzy subsets of R by $\Im(\Sigma, R)$.

Let A be a subset of a hemiring R. Then the interval valued characteristic function $\hat{C}_A$ of A is defined to be a function $\hat{C}_A : R \to \Sigma$ such that for all $x \in R$

$$\hat{C}_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

Clearly $if\ \hat{C}_A \in \Im(\Sigma, R)$. Note that $\hat{C}_R(x) = 1$ for all $x \in R$.

An interval valued fuzzy subset of R of the form

$$x_{\hat{t}}(y) = \begin{cases} \hat{t}(\neq 0) & if\ y = x \\ 0 & if\ y \neq x \end{cases}$$

is said to be interval valued fuzzy point with support x and value $\hat{t}$. An interval valued fuzzy point $x_{\hat{t}}$ is said to belong to (resp. be quasi-coincident with) interval valued fuzzy subset $\hat{\lambda}$, written as $x_{\hat{t}} \in \hat{\lambda}$ (resp. $x_{\hat{t}} q \hat{\lambda}$) if $\hat{\lambda}(x) \ge \hat{t}$ (resp. $\hat{\lambda}(x) + \hat{t} > 1$). $x_{\hat{t}} \in \vee q \hat{\lambda}$ means $x_{\hat{t}} \in \hat{\lambda}$ or $x_{\hat{t}} q \hat{\lambda}$ and $x_{\hat{t}} \in \wedge q \hat{\lambda}$ means $x_{\hat{t}} \in \hat{\lambda}$ and $x_{\hat{t}} q \hat{\lambda}$. $x_{\hat{t}} \overline{\alpha} \hat{\lambda}$ means that $x_{\hat{t}} \alpha \hat{\lambda}$ does not hold for $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$[cf. [14].

For any $\hat{\lambda}, \hat{\mu} \in \Im(\Sigma, R)$, union and intersection of $\hat{\lambda}$ and $\hat{\mu}$ are defined as, for all $x \in R$,

$$(\hat{\lambda} \vee \hat{\mu})(x) = [\lambda^-(x) \vee \mu^-(x), \lambda^+(x) \vee \mu^+(x)]$$

$$(\hat{\lambda} \wedge \hat{\mu})(x) = [\lambda^-(x) \wedge \mu^-(x), \lambda^+(x) \wedge \mu^+(x)].$$

Further for any $\hat{\lambda}, \hat{\mu} \in \Im(\Sigma, R)$, $\hat{\lambda} \le \hat{\mu}$ if and only if $\hat{\lambda}(x) \le \hat{\mu}(x)$, that is $\lambda^-(x) \le \mu^-(x)$ and $\lambda^+(x) \le \mu^+(x)$, for all $x \in R$.

3.1 Definition [23]

Let $\hat{\lambda}$, $\hat{\mu} \in \Im(\Sigma, R)$. Then the h-intrinsic product of $\hat{\lambda}$ and $\hat{\mu}$ is denoted and defined by

$$(\hat{\lambda} \odot \hat{\mu})(x) = \bigvee_{x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n a'_j b'_j + z} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^m \hat{\lambda}(a_i) \right) \wedge \left(\bigwedge_{i=1}^m \hat{\mu}(b_i) \right) \wedge \\ \left(\bigwedge_{j=1}^n \hat{\lambda}(a'_j) \right) \wedge \left(\bigwedge_{j=1}^n \hat{\mu}(b'_j) \right) \end{array} \right\}$$

for all $x \in R$, if x can be expressed as $x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n a'_j b'_j + z$, and 0 if x cannot be expressed as $x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n a'_j b'_j + z$.

3.2 Lemma [23]

Let R be a hemiring and $A, B \subseteq R$. Then we have

(i) $A \subseteq B$ if and only if $\hat{C}_A \le \hat{C}_B$.

(ii) $\hat{C}_A \wedge \hat{C}_B = \hat{C}_{A \cap B}$

(iii) $\hat{C}_A \odot \hat{C}_B = \hat{C}_{\overline{AB}}$

4. Interval Valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-Fuzzy *h*-Subhemiring and Interval Valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *h*-Ideals

4.1 Definition

Let $\hat{\lambda} \in \Im(\Sigma, R)$. Then $\hat{\lambda}$ is called an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-subhemiring of R if it satisfies:

1a. $(x+y)_{\min\{\hat{t},\hat{r}\}} \overline{\in} \hat{\lambda} \Rightarrow x_{\hat{t}} \overline{\in} \vee \overline{q} \hat{\lambda}$ or $y_{\hat{r}} \overline{\in} \vee \overline{q} \hat{\lambda}$,

2a. $(xy)_{\min\{\hat{t},\hat{r}\}} \overline{\in} \hat{\lambda} \Rightarrow x_{\hat{t}} \overline{\in} \vee \overline{q} \hat{\lambda}$ or $y_{\hat{r}} \overline{\in} \vee \overline{q} \hat{\lambda}$,

3a. $x + a + z = b + z$ and $x_{\min\{\hat{t},\hat{r}\}} \overline{\in} \hat{\lambda} \Rightarrow a_{\hat{t}} \overline{\in} \vee \overline{q} \hat{\lambda}$ or $b_{\hat{r}} \overline{\in} \vee \overline{q} \hat{\lambda}$,

for all $a, b, x, y, z \in R$ and $t, r \in (0,1]$.

4.2 Definition

Let $\hat{\lambda} \in \Im(\Sigma, R)$. Then $\hat{\lambda}$ is called an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) h-ideal of R if it satisfies $(1a), (3a)$ and

4a. $(yx)_{\hat{t}} \overline{\in} \hat{\lambda} \Rightarrow x_{\hat{t}} \overline{\in} \vee \overline{q} \hat{\lambda}$

(resp.5a. $(xy)_{\hat{t}} \overline{\in} \hat{\lambda} \Rightarrow x_{\hat{t}} \overline{\in} \vee \overline{q} \hat{\lambda}$),

for all $x, y \in R$ and $t \in (0,1]$.

An interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left and right h-ideal of R is called interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy h-ideal of R

4.3 Definition

Let $\hat{\lambda}\in\Im(\Sigma,R)$. Then $\hat{\lambda}$ is called an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior h-ideal of R if it satisfies $(1a),(2a),(3a)$ and

6a. $xyz_{\hat{t}}\,\overline{\in}\hat{\lambda}\Rightarrow y_{\hat{t}}\,\overline{\in}\vee\overline{q}\hat{\lambda},$

for all $x,y,z\in R$ and $t\in(0,1]$.

4.4 Definition

Let $\hat{\lambda}\in\Im(\Sigma,R)$, where R is a commutative hemiring with unity . Then $\hat{\lambda}$ is called an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy prime h-ideal of R if it satisfies $(1a),(3a),(4a),(5a)$ and

$(7a)$ $x_{\hat{t}}\,\overline{\in}\hat{\lambda}$ or $y_{\hat{r}}\,\overline{\in}\hat{\lambda}\Rightarrow(xy)_{\min\{\hat{t},\hat{r}\}}\,\overline{\in}\vee\overline{q}\hat{\lambda},$

for all $x,y\in R$ and $t,r\in(0,1]$.

4.5 Definition

Let $\hat{\lambda}\in\Im(\Sigma,R)$, where R is a commutative hemiring with unity . Then $\hat{\lambda}$ is called an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy semiprime h-ideal of R if it satisfies $(1a),(3a),(4a),(5a)$ and

$(8a)$ $x_{\hat{t}}\,\overline{\in}\hat{\lambda}\Rightarrow(x^2)_{\hat{t}}\,\overline{\in}\vee\overline{q}\hat{\lambda},$

for all $x\in R$ and $t\in(0,1]$.

4.6 Theorem

Let I be a non-empty subset of a hemiring R and $\hat{\lambda}\in\Im(\Sigma,R)$. We define:

$$\hat{\lambda}\;x\;=\begin{cases}<0.5 & \text{for } x\notin I\\ 1 & \text{otherwise}\end{cases}$$

Then $\hat{\lambda}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy h-subhemiring (h-ideal, interior h-ideal) of R if and only if I is an h-subhemiring (h-ideal, interior h-ideal) of R

4.7 Theorem

Let I be a non-empty subset of a commutative hemiring R with unity and $\hat{\lambda}\in\Im(\Sigma,R)$. We define:

$$\hat{\lambda}\;x\;=\begin{cases}<0.5 & \text{for } x\notin I\\ 1 & \text{otherwise}\end{cases}$$

Then $\hat{\lambda}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy prime h-ideal (resp. semiprime h-ideal) of R if and only if I is a prime h-ideal (resp. semiprime h-ideal) of R.

4.8 Theorem

For any $\hat{\lambda}\in\Im(\Sigma,R)$. (a through $(8a)$ are equivalent to $(1b)$ through $(8b)$ respectively, where

$(1b)$ $\max\{\hat{\lambda}(x+y),0.5\}\geq\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$,

$(2b)$ $\max\{\hat{\lambda}(xy),0.5\}\geq\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$,

$(3b)$ $x+a+z=b+z\Rightarrow\max\{\hat{\lambda}(x),0.5\}\geq\min\{\hat{\lambda}(a),\hat{\lambda}(b)\}$,

$(4b)$ $\max\{\hat{\lambda}(yx),0.5\}\geq\hat{\lambda}(x)$,

$(5b)$ $\max\{\hat{\lambda}(xy),0.5\}\geq\hat{\lambda}(x)$,

$(6b)$ $\max\{\hat{\lambda}(xyz),0.5\}\geq\hat{\lambda}(y)$,

$(7b)$ $\max\{\hat{\lambda}(x),\hat{\lambda}(y),0.5\}\geq\hat{\lambda}(xy)$,

$(8b)$ $\max\{\hat{\lambda}(x),0.5\}\geq\hat{\lambda}(x^2)$.

Proof We prove $(1a)$ is equivalent to $(1b)$. Other follows in an analogous way.

$(1a)\Rightarrow(1b)$ Suppose $(1b)$ does not hold. Then there exists $x,y\in R,$ such that $\max\{\hat{\lambda}(x+y),0.5\}<\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$. Then we can choose $t\in(0.5,1]$, such that $\max\{\hat{\lambda}(x+y),0.5\}<\hat{t}<\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$. This implies $(x+y)_{\hat{t}}\,\overline{\in}\hat{\lambda}$ but $x_{\hat{t}}\in\wedge q$ and $y_{\hat{t}}\in\wedge q$, which is a contradiction. So $(1b)$ holds.

$(1b)\Rightarrow(1a)$ Let $x,y\in R,$ and $t,r\in(0,1],$ be such that $(x+y)_{\min\{\hat{t},\hat{r}\}}\overline{\in}\hat{\lambda}$. Then $\hat{\lambda}(x+y)<\min\{\hat{t},\hat{r}\}$. If $\max\{\hat{\lambda}(x+y),0.5\}=\hat{\lambda}(x+y),$ then $\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}\leq\hat{\lambda}(x+y)<\min\{\hat{t},\hat{r}\}\Rightarrow\hat{\lambda}(x)<\hat{t}$ or $\hat{\lambda}(y)<\hat{r}\Rightarrow x_{\hat{t}}\,\overline{\in}\hat{\lambda}$ or $y_{\hat{r}}\,\overline{\in}\hat{\lambda}\Rightarrow x_{\hat{t}}\,\overline{\in}\vee\overline{q}\hat{\lambda}$ or $y_{\hat{r}}\,\overline{\in}\vee\overline{q}\hat{\lambda}$. If $\max\{\hat{\lambda}(x+y),0.5\}=0.5,$ then $\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}\leq0.5$. Suppose $x_{\hat{t}}\in\hat{\lambda}$ and

$y_{\hat{r}} \in \hat{\lambda}$, then $\hat{\lambda}(x) \geq \hat{t}$ and $\hat{\lambda}(y) \geq \hat{r}$. Thus $\hat{t} \leq \hat{\lambda}(x) < 0.5$ or $\hat{r} \leq \hat{\lambda}(y) < 0.5 \Rightarrow x_{\hat{t}} \overline{q} \hat{\lambda}$ or $y_{\hat{r}} \overline{q} \hat{\lambda} \Rightarrow x_{\hat{t}} \overline{\in} \vee \overline{q} \hat{\lambda}$ or $y_{\hat{r}} \overline{\in} \vee \overline{q} \hat{\lambda}$. This proves $(1a)$.

By using Theorem 4.8, the following results follow immediately.

4.9. Theorem

Let $\hat{\lambda} \in \Im(\Sigma, R)$. Then $\hat{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-subhemiring of R if and only if it satisfies $(1b), (2b)$ and $(3b)$.

4.10 Theorem

Let $\hat{\lambda} \in \Im(\Sigma, R)$. Then $\hat{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-ideal of R if and only if it satisfies $(1b)$, $3b$, $4b$ and $(5b)$.

4.11 Theorem

Let $\hat{\lambda} \in \Im(\Sigma, R)$. Then $\hat{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior h-ideal of R if and only if it satisfies $(1b)$, $2b$, $3b$ and $(6b)$.

4.12 Theorem

Let $\hat{\lambda} \in \Im(\Sigma, R)$, where R is a commutative hemiring with unity. Then $\hat{\lambda}$ is said to be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy prime h-ideal of R if and only if it satisfies $(1b), (3b), (4b), (5b)$ and $(7b)$.

4.13 Theorem

Let $\hat{\lambda} \in \Im(\Sigma, R)$, where R is a commutative hemiring with unity . Then $\hat{\lambda}$ is said to be an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy semiprime h-ideal of R if and only if it satisfies $(1b), (3b), (4b), (5b)$ and $(8b)$.

4.14. Theorem

A non-empty subset A of a hemiring R is an h-subhemiring (h-ideal, interior h-ideal) of R if and only if $\hat{C}_A$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-subhemiring (h-ideal, interior h-ideal) of R.

4.15 Theorem

A non-empty subset A of a commutative hemiring R with unity is a prime h-ideal (resp. semiprime h-ideal) of R if and only if $\hat{C}_A$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy prime h-ideal (resp. semiprime h-ideal) of R.

4.16 Theorem

Every interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-ideal of a hemiring R is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior h-ideal of R.

4.17 Remark

Converse of the Theorem 4.16 is not true in general.

4.18 Example

Consider the hemiring $R = \{0, a, b, c\}$ with the following Cayley tables

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Let $\hat{\lambda} \in \Im(\Sigma, R)$ defined by $\hat{\lambda}(0) = \hat{\lambda}(a) = [0.505, 0.605], \hat{\lambda}(b) = \hat{\lambda}(c)$ $= [0.205, 0.305]$. Then $\hat{\lambda}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior h-ideal of R, but it is not an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-ideal of R.

4.19 Theorem

Every interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy prime h-ideal of a hemiring R is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy semiprime h-ideal of R.

4.20 Remark

Converse of the Theorem 4.19 is not true in general.

4.21 Example

Let $N_0 = \{0\} \cup N$ and $p_1, p_2, p_3, ...$ be the distinct prime numbers in N_0. If $J^0 = N_0$ and $J^l = p_1 p_2 p_3 ... p_l N_0$, $l = 1, 2, 3, ...$, then $J^0 \supset J^1 \supset J^2 \supset ....J^n \supset J^{n+1} \supset ....$ As every non-zero element of N_0 has unique prime factorization, J^l is a semiprime h-ideal for $l = 2, 3, ...$ but not a prime h-ideal. Then for such values of l, by Theorem 4.15, $\hat{C}_{J^l}$ is an interval valued $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy semiprime h-ideal of *R*, but $\hat{C}_{J'}$ is not an interval valued $(\bar{\in},\bar{\in}\vee\bar{q})$-fuzzy prime h-ideal of *R*.

5. *h*-hemiregular and *h*-semisimple hemirings

In this section we characterize *h*-hemiregular and *h*-semisimple hemirings by the properties of their interval valued $(\bar{\in},\bar{\in}\vee\bar{q})$-fuzzy *h*-ideals and interval valued $(\bar{\in},\bar{\in}\vee\bar{q})$-fuzzy interior *h*-ideals.

5.1 Definition [27]

A hemiring *R* is said to be h-hemiregular if for each $x\in R,$ there exist $a,b,z\in R$ such that $x+xax+z=xbx+z$.

5.2 Lemma [27]

A hemiring *R* is *h*-hemiregular if and only if for any right *h*-ideal *I* and any left *h*-ideal *L* of *R* we have $\overline{IL}=I\cap L$.

5.3 Lemma [26]

A hemiring *R,* is *h*-semisimple if and only if one of the following holds:

i. For all $x\in R,$ there exists $c_i,d_i,e_i,f_i,c_j',d_j',e_j',f_j'\in R$ such that $x+\sum_{i=1}^{m}c_ixd_ie_ixf_i+z=\sum_{j=1}^{n}c_j'xd_j'e_j'xf_j'+z.$

ii. For all $x\in R, x\in\overline{RxRxR}.$

iii. For all $A\subseteq R, A\subseteq\overline{RARAR}.$

5.4 Definition

Let $\hat{\lambda},\hat{\mu}\in\Im(\Sigma,R)$. Then we say $\hat{\lambda}[\bar{\in}\vee\bar{q}]\hat{\mu},$ if $x_{\hat{t}}\bar{\in}\hat{\mu}\Rightarrow x_{\hat{t}}\bar{\in}\vee\bar{q}\hat{\lambda}.$

5.5 Theorem

Let $\hat{\lambda},\hat{\mu}\in\Im(\Sigma,R)$. Then $\hat{\lambda}[\bar{\in}\vee\bar{q}]\hat{\mu},$ if and only if $\max\{\hat{\mu}(x),0.5\}\geq\hat{\lambda}(x),$ for all $x\in R.$

Proof. Let us assume $\hat{\lambda}[\bar{\in}\vee\bar{q}]\hat{\mu}.$ To prove $\max\{\hat{\mu}(x),0.5\}\geq\hat{\lambda}(x).$ Suppose on the contrary, there exists $x\in R,$ such that $\max\{\hat{\mu}(x),0.5\}<\hat{\lambda}(x).$ Then for some $t,$ $\max\{\hat{\mu}(x),0.5\}<\hat{t}<\hat{\lambda}(x).$ Then $\hat{\mu}(x)<\hat{t},0.5<\hat{t},\hat{\lambda}(x)>\hat{t}\Rightarrow x_{\hat{t}}\bar{\in}\hat{\mu}$ and $x_{\hat{t}}\in\wedge q\hat{\lambda}.$ Which is a contradiction, so $\max\{\hat{\mu}(x),0.5\}\geq\hat{\lambda}(x),$ for all $x\in R.$

Conversely, assume $\max\{\hat{\mu}(x),0.5\}\geq\hat{\lambda}(x),$ for all $x\in R.$ To prove $\hat{\lambda}[\bar{\in}\vee\bar{q}]\hat{\mu}.$ Let $x_{\hat{t}}\bar{\in}\hat{\mu}.$ Then $\hat{\mu}(x)<t.$ Now $\hat{\lambda}(x)\leq\max\{\hat{\mu}(x),0.5\}<\max\{t,0.5\}.$

If $\max\{t,0.5\}=t,$ then $\hat{\lambda}(x)<t\Rightarrow x_{\hat{t}}\bar{\in}\hat{\lambda}.$

If $\max\{t,0.5\}=0.5,$ then $\hat{\lambda}(x)+\hat{t}<0.5+0.5=1\Rightarrow x_{\hat{t}}\bar{q}\hat{\lambda}.$ Hence $x_{\hat{t}}\bar{\in}\vee\bar{q}\hat{\lambda}.$

5.6 Definition

Let $\hat{\lambda},\hat{\mu}\in\Im(\Sigma,R).$ Then we say $\hat{\lambda}\sim\hat{\mu},$ if and only if $\hat{\lambda}[\bar{\in}\vee\bar{q}]\hat{\mu}$ and $\hat{\mu}[\bar{\in}\vee\bar{q}]\hat{\lambda}.$

5.7 Lemma

The relation $"\sim"$ on $\Im(\Sigma,R)$ is an equivalence relation.

5.8 Lemma

Let $\hat{\lambda}\in\Im(\Sigma,R).$ Then $\hat{\lambda}$ satisfies conditions $(1b)$ and $(3b)$ if and only if it satisfies condition

$(9b)$ $(\hat{\lambda}+\hat{\lambda})[\bar{\in}\vee\bar{q}]\hat{\lambda}\vee 0.5.$

Proof. Suppose $\hat{\lambda}$ satisfies conditions $(1b)$ and $(3b)$. Let $x\in R.$ Then

$$\begin{aligned}
(\hat{\lambda}+\hat{\lambda})(x) &\leq (\hat{\lambda}+\hat{\lambda})(x)\vee 0.5\\
&=\left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z}\hat{\lambda}(a_1)\wedge\hat{\lambda}(a_2)\wedge\hat{\lambda}(b_1)\wedge\hat{\lambda}(b_2)\right)\vee 0.5\\
&=\left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z}(\hat{\lambda}(a_1)\wedge\hat{\lambda}(b_1))\wedge(\hat{\lambda}(a_2)\wedge\hat{\lambda}(b_2)\right)\vee 0.5\\
&\leq\left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z}\left\{\begin{matrix}\max(\hat{\lambda}(a_1+b_1),0.5)\wedge\\ \max(\hat{\lambda}(a_2+b_2),0.5)\end{matrix}\right\}\right)\vee 0.5\\
&\leq\left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z}\left\{(\hat{\lambda}(a_1+b_1)\wedge\hat{\lambda}(a_2+b_2))\vee 0.5)\right\}\right)\vee 0.5\\
&\leq\hat{\lambda}(x)\vee 0.5.
\end{aligned}$$

Thus $(\hat{\lambda}+\hat{\lambda})[\bar{\in}\vee\bar{q}]\hat{\lambda}\vee 0.5$. So $(9b)$ is satisfied.

Converse is straightforward.

5.9 Theorem

Let $\hat{\lambda}\in\Im(\Sigma,R).$ Then $\hat{\lambda}$ is an interval valued $(\bar{\in},\bar{\in}\vee\bar{q})$-fuzzy left (resp. right) h-ideal of R if and only if $\hat{\lambda}$ satisfies conditions

9b $(\hat{\lambda}+\hat{\lambda})[\overline{\in}\vee\overline{q}]\hat{\lambda}\vee 0.5$

10b $(\hat{\mathbf{R}}\odot\hat{\lambda})[\overline{\in}\vee\overline{q}]\hat{\lambda}\vee 0.5$ resp. $(\hat{\lambda}\odot\hat{\mathbf{R}})[\overline{\in}\vee\overline{q}]\hat{\lambda}\vee 0.5$.

Proof. By using Lemma 5.8, proof of the Theorem is straightforward.

5.10 Lemma

If $\hat{\lambda}$ and $\hat{\mu}$ are interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right and left h-ideals of R respectively, then $(\hat{\lambda}\odot\hat{\mu})[\overline{\in}\vee\overline{q}](\hat{\lambda}\wedge\hat{\mu})$.

5.11 Theorem

For a hemiring R the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $(\hat{\lambda}\wedge\hat{\mu})\sim(\hat{\lambda}\odot\hat{\mu})$ for every interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right h-ideal $\hat{\lambda}$ and every interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left h-ideal $\hat{\mu}$ of R.

Proof. $i \Rightarrow ii$: Let $\hat{\lambda}$ be an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right h-ideal and $\hat{\mu}$ be an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left h-ideal of R. Then by Lemma 5.10, $(\hat{\lambda}\odot\hat{\mu})[\overline{\in}\vee\overline{q}](\hat{\lambda}\wedge\hat{\mu})$. Let $x\in R$. Then there exist $a_1,a_2,z\in R$ such that $x+xa_1x+z=xa_2x+z$. Thus

$$\hat{\lambda}\odot\hat{\mu}\ \ x\ \vee 0.5$$

$$=\left(\bigvee_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}c_jd_j+z}\left(\begin{array}{c}\left(\bigwedge_{i=1}^{m}\hat{\lambda}\ a_i\right)\wedge\left(\bigwedge_{i=1}^{m}\hat{\mu}\ b_i\right)\wedge\\ \left(\bigwedge_{j=1}^{n}\hat{\lambda}\ c_j\right)\wedge\left(\bigwedge_{j=1}^{n}\hat{\mu}(d_j)\right)\end{array}\right)\right)\vee 0.5$$

$$\geq\{(\hat{\lambda}\ x\ \wedge\hat{\mu}\ a_1x\ \wedge\hat{\mu}\ a_2x\ )\vee 0.5\}$$

$$\geq\{(\hat{\lambda}\ x\ \wedge(\hat{\mu}\ a_1x\ \vee 0.5)\wedge(\hat{\mu}\ a_2x\ \vee 0.5))\}$$

$$\geq\ \hat{\lambda}\ x\ \wedge\hat{\mu}\ x$$

$$=\left(\hat{\lambda}\wedge\hat{\mu}\right)\left(x\right).$$

$$\Rightarrow(\hat{\lambda}\wedge\hat{\mu})[\overline{\in}\vee\overline{q}](\hat{\lambda}\odot\hat{\mu})$$

Hence $(\hat{\lambda}\wedge\hat{\mu})\sim(\hat{\lambda}\odot\hat{\mu})$.

$(ii)\Rightarrow(i)$ Let I be a right and J be a left h-ideal of R. Then by Theorem 4.14, $\hat{C}_I$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right h-ideal and $\hat{C}_J$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left h-ideal of R. Then by hypothesis and by Lemma 3.2,

$$\hat{C}_{\overline{IJ}}=\hat{C}_I\odot\hat{C}_J\sim\hat{C}_I\wedge\hat{C}_J=\hat{C}_{I\cap J}.$$

$$\Rightarrow\overline{IJ}=I\cap J.$$

Hence by Lemma 5.2, R is h-hemiregular.

5.12 Theorem

Let R be an h-semisimple hemiring and $\hat{\lambda}\in\Im(\Sigma,R)$. Then $\hat{\lambda}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy h-ideal of R if and only if it is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior h-ideal of R.

Proof. It is obvious by the definition that an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy h-ideal of R is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior h-ideal of R. Conversely assume $\hat{\lambda}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior h-ideal of R. Let $x,y\in R$. Then by Lemma 5.3, there exists $c_i,d_i,e_i,f_i,c_j',d_j',e_j',f_j'\in R$ such that

$x+\sum_{i=1}^{m}c_ixd_ie_ixf_i+z=\sum_{j=1}^{n}c_j'xd_j'e_j'xf_j'+z.$ Which implies

$$xy+\sum_{i=1}^{m}c_ixd_ie_ixf_iy+zy=\sum_{j=1}^{n}c_j'xd_j'e_j'xf_j'y+zy.$$

Thus

$$\max\ \hat{\lambda}(xy),0.5$$

$$\geq\min\left\{\hat{\lambda}\left(\sum_{i=1}^{m}c_ixd_ie_ixf_iy\right),\hat{\lambda}\left(\sum_{j=1}^{n}c_j'xd_j'e_j'xf_j'y\right)\right\}\vee 0.5$$

$$\geq\hat{\lambda}(x)$$

Thus $\hat{\lambda}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right h-ideal of R. Similarly $\hat{\lambda}$ is an interval valued $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left h-ideal of R.

5.13 Theorem

The following conditions about a hemiring R are equivalent:

(i) R is *h*-semisimple.

(ii) For any interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy interior *h*-ideals $\hat{\lambda}$ and $\hat{\mu}$ of $R,$ $\hat{\lambda} \wedge \hat{\mu} \sim \hat{\lambda} \square \hat{\mu}.$

Proof. $(i) \Rightarrow (ii)$ Let $\hat{\lambda}, \hat{\mu}$ be interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy interior *h*-ideals of $R.$ Then by Theorem 5.12, $\hat{\lambda}, \hat{\mu}$ are interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy *h*-ideals of $R.$ Then by Lemma 5.10, $(\hat{\lambda} \square \hat{\mu})[\bar{\in} \vee \bar{q}](\hat{\lambda} \wedge \hat{\mu}).$ Now as R is *h*-semisimple, so for any $x \in R,$ there exists $c_i, d_i, e_i, f_i, c_j^{'}, d_j^{'}, e_j^{'}, f_j^{'} \in R$, such that $x + \sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c_j^{'} x d_j^{'} e_j^{'} x f_j^{'} + z.$

Now

$$\max \ (\hat{\lambda} \square \hat{\mu})(x), 0.5$$

$$= \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j^{'} b_j^{'} + z} \left[\begin{array}{c} \left(\bigwedge_{i=1}^{m} \hat{\lambda} \ a_i \right) \wedge \left(\bigwedge_{i=1}^{m} \hat{\mu} \ b_i \right) \wedge \\ \left(\bigwedge_{j=1}^{n} \hat{\lambda} \ a_j^{'} \right) \wedge \left(\bigwedge_{j=1}^{n} \hat{\mu} \ b_j^{'} \right) \end{array} \right] \vee 0.5$$

$$\geq \min\{\hat{\lambda}(c_i x d_i), \hat{\lambda}(c_j^{'} x d_j^{'}), \hat{\mu}(e_i x f_i), \hat{\mu}(e_j^{'} x f_j^{'})\} \vee 0.5$$

$$\geq \min\{\hat{\lambda}(x), \hat{\mu}(x)\} = (\hat{\lambda} \wedge \hat{\mu})(x)$$

$\Rightarrow (\hat{\lambda} \wedge \hat{\mu})[\bar{\in} \vee \bar{q}](\hat{\lambda} \square \hat{\mu}).$ Hence $\hat{\lambda} \wedge \hat{\mu} \sim \hat{\lambda} \square \hat{\mu}.$

$(ii) \Rightarrow (i)$ Let A be an *h*-ideal of *R*. Then A is an interior *h*-ideal of *R*. Then by Theorem 4.14, $\hat{C}_A$ is an interval valued $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy interior *h*-ideal of *R*. Then

$$\hat{C}_A = (\hat{C}_A \wedge \hat{C}_A) \sim \hat{C}_A \square \hat{C}_A = \hat{C}_{\overline{AA}} = \hat{C}_{\overline{A^2}}.$$

$$\Rightarrow \overline{A^2} = A.$$

Hence *R*. is h-semisimple.

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