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# ON THE UNIFORMLY CONVERGENCE SPECTRAL EXPANSIONS CONNECTED WITH SCHRÖDINGER'S OPERATOR OF CONTINUOUS FUNCTIONS IN A CLOSED DOMAIN

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Present paper is devoted to study summability problem of spectral expansions in a closed domain. We consider here as a spectral expansions eigenfunction expansions connected with one Schrodinger's operator with singular potential in two dimensional domains with smooth boundary.

**Keywords**: Schrödinger's operator, Spectral expansions, Summability in a closed domain.

## 1. Introduction

One of very important operators in quantum mechanics is Schrödinger's operator with singular potential. This operator is acting on the functions belonging to the Hilbert space [1]. Operators in infinite dimensional Hilbert space are not defined for all the functions of the space. That's why we should define the domains of the operator. In the present paper as a domain, we consider appropriate Sobolev spaces [2]. The reason of importance of the domain is operators in quantum mechanics need to be self-adjoint [3]. For self-adjoint operator, eigenvalues are real and the eigenfunctions form a complete set of orthogonal functions so that any function of the Hilbert space of the system can be expanded in this set in strong (norm) topology. When the Hamiltonian H of the system is a self-adjoint operator, then the time-dependent Schrodinger equation has a unique solution. This solution will be presented by spectral expansions connected with corresponding Hamiltonian. That is why it is very important to prove applicability of the spectral expansions method for the presentation solutions of the problems modern mathematical physics. Thus one has to study convergence and/or summability of such spectral expansions. In case of free Hamiltonian corresponding methods and theory developed by many scientists [4, 5].

Let $\Omega$ is a bounded domain in $R^2$ with smooth boundary $\partial\Omega$. We will consider potential function $q(x)$ as a positive function from Sobolev's space $W_2^1(\Omega)$ with singularities at a point $x_0 \in \Omega$ (or in finite numbers of points) and enough smooth out of this point. Also, we will suppose that first eigenvalue problem for the corresponding Schrödinger's operator with potential $q$ produce selfadjoint operator so the problem has countable number eigenfunctions $u_n(x)$ answering to the eigenvalues $\lambda_n$.

Problems of convergence and summability eigenfunction expansions in compact subsets of the domain were studied by many mathematicians. Significant contributions, for solving these problems, are given by Russian mathematical schools, as in Moscow University scientists headed by Prof. Il'in V.A. However, problems of convergences and summability of eigenfunction expansions near the boundary of the domain creates significant difficulties. The last statement was mentioned in [5] and it was also formulated a problem for the convergence and summability in a closed domain. Further this problem was investigated by E.I. Moiseev [6] and he obtains an

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estimation of a sum of eigenfunctions of first boundary value problem for the Laplace operator in a closed domain. A.A. Rakhimov and Kamran Zakaria in [7] obtained the same estimation in case of Schrodinger's operator.

In present paper, we prove a theorem on summability of eigenfunction expansions connected with Schrodinger's operator in a class of continuous functions. In this we use an estimation of eigenfunctions obtained in [7]. The same problem for the free Hamiltonian operator was solved by A. A. Rakhimov in [8]. Eigenfunction expansions connected with Schrodinger's operator in compact subsets of the domain studied by A.R. Khalmukhamedov [9] in various functional spaces and by A.A. Rakhimov [10] in a class of distributions.

## 2. Definitions and Main Result

Define Reisz means of partial sums eigenfunction expansions by system $\{u_n(x)\}$:

$$E_\lambda^s f(x) = \sum_{\lambda_n \le \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x), \tag{1}$$

where $f_n = (f, u_n)$ and $\{\lambda_n\}$- a sequence of eigenvalues: $0 < \lambda_1 \le \lambda_2 \le ................ \le \lambda_n \to \infty$.

Main result of the present work is to prove following theorem.

### *2.1. Theorem*

Let $f$ a finite and continuous in a domain $G$ function. Then Reisz means (1) of order $s$, $s > \frac{N-1}{2}$, convergence to $f$ uniformly in a closed domain $\overline{G}$.

### *2.2. Preliminaries and Auxiliary Lemmas.*

For any number $h > 0$ by $G_h$ denote following set

$$G_h = \{x \in G : \text{dist}(x, \partial G) > h\}.$$

Let $x \in G_h$ and $y \in \overline{G}$. Consider a function of variable $r = |x-y|$:

$$V(r) = \begin{cases} \Gamma(s+1)\; 2^s\; (2\pi)^{\frac{-N}{2}}\; \lambda^{\frac{N}{4}-\frac{s}{2}}\; \dfrac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{r^{\frac{n}{2}+s}}, & r \le R \\ 0, & R > 0 \end{cases} \tag{2}$$

where R less $\frac{h}{4}$, $J_v(t) -$ Bessel's function of order $\nu$.

For eigenfunctions $u_n(x)$ we have following mean value formula in a ball $\{r \le R\}$ with the centre at $x \in G_h$:

$$S_t(u_n) = (2\pi)^{N/2} J_\beta(r\sqrt{\lambda_n})(r\sqrt{\lambda_n})^{-\beta} u_n(x) + \frac{\pi}{2} r^{-\beta} \int_0^r \left\{ \begin{matrix} J_\beta(t\sqrt{\lambda_n}) Y_\beta(r\sqrt{\lambda_n}) - \\ Y_\beta(t\sqrt{\lambda_n}) J_\beta(r\sqrt{\lambda_n}) \end{matrix} \right\} t^{\beta+1} S_t(qu_n)dt \tag{3}$$

Where $\beta = \frac{N-2}{2}$, $S_t(g)(x) = \int_\theta g(x + t\theta)d\theta$, and $J_v(t)$, $Y_v(t) -$ are Bessel's functions of order $\nu$.

Note that

$$\int_0^\infty J_{a+s}(\sqrt{\lambda}\; t) J_{a-1}\left(\sqrt{\lambda_k}\; t\right) t^{-s} dt = \int_0^\infty \begin{cases} \dfrac{\left(1 - \frac{\lambda_k}{\lambda}\right)^s \lambda^s \lambda_k^{\frac{a-1}{2}}}{2^s \Gamma(s+1) \lambda^{\frac{a+s}{2}}}, & \lambda_k \le \lambda \\ 0, & \lambda_k > \lambda \end{cases} \tag{4}$$

Using (3) we obtain following expression for Fourier coefficient of function $\overset{\lambda}{v}(|x-y|)$:

$$\overset{\lambda}{v}_n(x) = 2^s \Gamma(s+1) \lambda_n^{\frac{2-N}{4}} \lambda^{\frac{N-2s}{4}} u_n(x) \cdot \int_0^R J_{\frac{N}{2}+s}(\sqrt{\lambda} r) J_{\frac{N}{2}-1}(\sqrt{\lambda_n} r) r^{-s} dr +$$

$$+\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}}\cdot\int_0^R\left(r\sqrt{\lambda}\right)^{-\frac{N}{2}-s}J_{\frac{N}{2}+s}\left(r\sqrt{\lambda}\right)\cdot$$
$$r^{N-1-\beta}\cdot\int_0^r W_\beta(t,r,\sqrt{\lambda_n})\cdot t^{\beta+1}\cdot S_t(q\cdot u_n)dt \qquad (5)$$

Where $W_\beta(t,r,\sqrt{\lambda_n})=J_\beta(t\sqrt{\lambda_n})Y_\beta(r\sqrt{\lambda_n})-Y_\beta(t\sqrt{\lambda_n})J_\beta(r\sqrt{\lambda_n})$.

In right side of (5), divide first integral into two part as $\int_0^\infty - \int_R^\infty$

And taking into consideration (4) obtain following formula for $\overset{\lambda}{v}_n(x)$:

$$^\lambda v_n(x)=\delta_n^\lambda u_n(x)(1-\frac{\lambda_n}{\lambda})^s-2^s\Gamma(s+1)$$
$$\lambda_n^{\frac{1-N}{4}}\lambda^{\frac{N-1}{4}-\frac{s}{2}}u_n(x)I_1(\lambda,\lambda_n)+\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}} \qquad (6)$$
$$\frac{\pi}{2}\lambda^{\frac{N}{2}}\cdot I_2(\lambda,u_n),$$

where

$$I_1(\lambda,\lambda_n)=(\lambda\cdot\lambda_n)^{1/4}\int_R^\infty J_{\frac{N}{2}+s}(r\sqrt{\lambda})J_{\frac{N}{2}-1}(r\sqrt{\lambda_n})r^{-s}dr$$

$$\delta_n^\lambda=\begin{cases}1, & \lambda_n<\lambda\\ 0, & \lambda_n\geq\lambda\end{cases},$$

and

$$I_2(\lambda,u_n)=\cdot\int_0^R\left(r\sqrt{\lambda}\right)^{-\frac{N}{2}-s}J_{\frac{N}{2}+s}\left(r\sqrt{\lambda}\right)\cdot r^{N-1-\beta}\cdot$$
$$\int_0^r W_\beta(t,r,\sqrt{\lambda_n})\cdot t^{\beta+1}\cdot S_t(q\cdot u_n)dt \qquad (7)$$

Multiply both side of (6) to $u_n(x)$ and take summation by all numbers. As a result obtain following equality in sense of $L_2$ by $y$ :

$$\overset{\lambda}{v}(|x-y|)=\Theta^s(x,y,\lambda)-2^s\Gamma(s+1)\,\lambda^{\frac{N-1}{4}-\frac{s}{2}}$$
$$\sum_{n=1}^\infty u_n(x)\cdot u_n(y)\cdot\lambda_n^{\frac{1-n}{4}}\cdot I_1(\lambda,\lambda_n) \qquad (8)$$
$$+\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}}\cdot\sum_{n=1}^\infty u_n(y)\cdot I_2(\lambda,u_n)$$

where

$$\Theta^s(x,y,\lambda)=\sum_{\lambda_n<\lambda}\left(1-\frac{\lambda_n}{\lambda}\right)^s\cdot u_n(x)\cdot u_n(y)$$

is called Reisz means of spectral function.

Left side of (8) in $x\in G_h$ and $y\in\overline{G}$, denote by $V(x,y,\lambda)$.

In [7] it is obtained following uniformly by $y\in\overline{G}$ estimation for eigenfunctions of first boundary problem for Schrodinger's operator:

$$\sum_{|\sqrt{\lambda_n}-\mu|\leq 1}u_n^2(y)=O(\mu\ \ \ln^2\mu) \qquad (9)$$

From (9) it follows that for any positive number $\varepsilon$ it is valid estimation:

$$\sum_{\lambda_n<\lambda}u_n^2(y)\cdot\lambda_n^{\varepsilon-1}=O(\lambda^\varepsilon\cdot\ln^2\lambda) \qquad (10)$$

$$\sum_{\lambda_n>\lambda}u_n^2(y)\lambda_n^{-\varepsilon-1}=O(\lambda^{-\varepsilon}\cdot\ln^2\lambda) \qquad (11)$$

For integral $I_1(\lambda,\lambda_n)$ defined by equality (2.8) we have following estimation (see in [5]):

$$|I_1(\lambda,\lambda_n)|\quad\leq\quad\frac{c}{1+\left|\sqrt{\lambda_n}-\sqrt{\lambda}\right|} \qquad (12)$$

Let $f\in L_2(G)$ and $f_n$ its Fourier coefficients. Then from estimations (10) and (11) it follows that series

$$\sum_{n=1}^\infty f_n\cdot u_n(y)\cdot\lambda_n^{\frac{-1}{2}}\cdot I_1(\lambda,\lambda_n),$$

converges uniformly in closed domain $\overline{G}$. Therefore, for any function f from $L_2(G)$ integral

$$\int_G f(x)\ V(x,y,\lambda)\ dx$$

is continuous by $y \in \overline{G}$.

Let's support of a function $f(x) \in L_2(G)$ is in $G_h$. Then by definition for Reisz means of partial sum of Fourier series of function $f(x)$ by system $\{u_n(x)\}$, we will have

$$E_\lambda^S f(y) = \int_{G_h} f(x)\ V(x,y,\lambda)\ dx + 2^S\Gamma(s+1)$$
$$\lambda^{\frac{N-1}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} f_n \lambda_n^{\frac{1-N}{4}}\ u_n(y)\, I_1(\lambda,\lambda_n) + \tag{13}$$
$$\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}} \cdot \sum_{n=1}^{\infty} f_n I_2(\lambda,u_n)$$

Denote by $B(R,y)$ a ball of radius R with the center at point $y \in \overline{G}$. Then taking into account continuity of function

$$\int_{G_h} f(x)\ \overset{\lambda}{v}(r)\ dx = 2^s\Gamma(s+1)\ (2\pi)^{-\frac{N}{2}}\ \lambda^{\frac{N}{4}-\frac{s}{2}} \times$$
$$\times \int_{G_h \cap B(R,y)} f(x) J_{\frac{N}{2}+s}(\sqrt{\lambda}r)\cdot r^{-(\frac{N}{2}+s)}\ dr$$

and also fact that it is equal to first term in first part equality (13), obtain

$$E_\lambda^s f(y) = \int_{G_h} f(x)\ \overset{\lambda}{v}(|x-y|)\ dx + 2^s\cdot\Gamma(s+1)\cdot$$
$$\lambda^{\frac{N-1}{4}-\frac{s}{2}} \cdot \sum_{n=1}^{\infty} f_n\ \lambda_n^{\frac{1-N}{4}}\ u_n(y)\, I_1(\lambda,\lambda_n) + \tag{14}$$
$$\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}} \cdot \sum_{n=1}^{\infty} f_n I_2(\lambda,u_n)$$

*2.3. Lemma 1*

Let $y \in \overline{G}$. Then uniformly by y we have inequality

$$\sum_{n=1}^{\infty} u_n^2(y)\ \lambda_n^{\frac{1-N}{2}}\ [I_1(\lambda,\lambda_n)]^2 \le C\cdot \ell n^2\lambda\ . \tag{15}$$

**Proof** From (10) and (12) taking as $\lambda = 1$ it follows

$$\sum_{\sqrt{\lambda_\tau}\le 1} \lambda_n^{\frac{1-N}{2}}\ [I_1(\lambda,\lambda_n)]^2\ u_n^2(y) = O\left(\frac{1}{\lambda}\right).$$

Also from (12) , (11) and (10) obtain following estimations:

$$\sum_{1\le\sqrt{\lambda_n}\le\frac{\sqrt{\lambda}}{2}} \lambda_n^{\frac{1-N}{2}} \left[I_1(\lambda,\lambda_n)\right]^2\ u_n^2(y) = O\left(\frac{\ell n^2\lambda}{\sqrt{\lambda}}\right) \tag{16}$$

$$\sum_{\frac{\sqrt{3\lambda}}{2}\le\sqrt{\lambda_n}} \lambda_n^{\frac{1-N}{2}} \left[I_1(\lambda,\lambda_n)\right]^2 u_n^2(y) = O\left(\frac{\ell n^2\lambda}{\sqrt{\lambda}}\right). \tag{17}$$

For estimating term which answer to the numbers $n$ for which $\frac{\sqrt{\lambda}}{2} \le \sqrt{\lambda_n} \le \frac{\sqrt{3\lambda}}{2}$ we use (9) and (12). Denote by *k* least number for which $2^x \ge \frac{\sqrt{\lambda}}{2}$. Then taking into account (9) and (12) obtain

$$\sum_{\left|\sqrt{\lambda_n}-\sqrt{\lambda}\right| \le \frac{\sqrt{\lambda}}{2}} \lambda_n^{\frac{1-N}{2}} \left[I_1(\lambda,\lambda_n)\right]^2 u_n^2(y) \le$$
$$\sum_{m=1}^{k} \sum_{2^{m-1} \le \left|\sqrt{\lambda}-\sqrt{\lambda_n}\right| \le 2^m} \lambda_n^{\frac{1-N}{2}}\ u_n^2(y)\ 4^{1-m} \le c\cdot \ell n^2\lambda\ .$$

Lemma 1 proved

*2.4. Lemma 2.*

Let function $f(y)$ continuous and finite in G. If $s > (N-1)/2$, then uniformly by $y \in \overline{G}$ following inequality is valid

$$\left|E_\lambda^s f(y)\right| \le c\ \|f\|_\infty\ . \tag{18}$$

**Proof** : For proof inequality (18) first estimate each of terms in right side of (13). For estimation of first term in (13) we use following estimations for Bessel's function

$$|J_\nu(t)| \le \begin{cases} t^{-1/2}, & t \ge 1 \\ t^{\nu}, & t \le 1 \end{cases}.$$

Then dividing integral in right side of (13) into two obtain

$$\left|\int_{G_n} f(x)\cdot \overset{\lambda}{v}(r)dx\right| \le c_1 \cdot \|f\|_\infty$$
$$\left[\int_0^{1/\sqrt{\lambda}} r^{N-1}\cdot\left|\overset{\lambda}{v}(r)\right|dx + \int_{1/\sqrt{\lambda}}^{R} r^{N-1}\cdot\left|\overset{\lambda}{v}(r)\right|dr\right].$$

It is clear that from estimation of Bessel's function it follows that quantity in quadratic brackets bounded. Now estimate second term in right side of equality (13). For this we apply to following sum.

$$\sum_{n=1}^{\infty} f_n \cdot u_n(y)\cdot I_1(\lambda,\lambda_n)\ \lambda_n^{\frac{1-N}{4}}$$

Holder's inequality and Parsevvall's equality.

Then from lemma 1 it follows

$$\left|\sum_{n=1}^{\infty} f_n \cdot u_n(y)\cdot I_1(\lambda,\lambda_n)\cdot \lambda_n^{\frac{1-N}{4}}\right| \le c\cdot \ell n\lambda\ \|f\|_\infty$$

Third term in left side of (13) can be proved as second term. Lemma 2 is proved.

### *2.5. Proof of the theorem*

For any function from the space $C_0^\infty(G)$ convergence of Reisz means $E_\lambda^s f(y)$ will be uniformly in closed domain for any $s \ge 0$. Due to denseness of $C_0^\infty(G)$ in the space of continuous function on $G$, function $f(y)$ which satisfies conditions of the theorem can be approximated by functions from $C_0^\infty(G)$, which has supports in $G_h$, where positive number *h* depends only from distance between support of the function $f(y)$ and boundary of the domain $G$. Then statement of the theorem follows from lemma 2. Theorem is proved.

## 3. Remarks

Main result of the present paper supplies applicability of the spectral expansions connected with given Hamiltonian for representation unique solution of the time dependent Schrodinger's equation in a closed domain. Formerly this result was known only for free Hamiltonian systems [8] and for singular operators only in compact subsets of the domain [9, 10].

Methods used in the paper and its results can be used in investigations of the solvability of the problems in quantum mechanics, nuclear physics and mathematical physics.

## References


[1] M. Reed and B. Simon, Methods of Modern Mathematical Physics, I. Functional Analysis, New York, London (1972) p.357.

[2] E.B. Davies, Spectral theory and differential operators, Cambridge University Press (1995) p. 188.

[3] M. Reed, B. Simon, Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-adjoiness, Academic Press New York London (1978) p.394.

[4] Sh. A. Alimov, V.A. Il'in and E.M. Nikishin, Convergence, Uspekhi Mat. Nauk, **31**:6(192) (1976) 28.

[5] V.A. Il'in, Spectral Theory of Differential Operators, Moscow, Nauka (1991) 367.

[6] E.I. Moiseev, Soviet Math. Dokl **18** (1977) 549.

[7] A.A. Rakhimov, Z. Kamran, On an Estimation of Eigenfunctions of Schrödinger's Operatorin a Closed Domain, MathDigest, UPM, Malaysia, June Edition (2010) p. 23-27.

[8] A.A. Rakhimov, J. Izvestiya of Uzbek Academy of Science **N 6** (1987) 17.

[9] A.R. Khalmukhamedov, Spectral Theory of Elliptic Equations with Singular Coefficients, Doctoral Thesis, Tashkent (1998) p.1-205.

[10] A.A. Rakhimov, J. Astrophysics and Applied Mathematics **V1** (2009) 100.